1. (i−1)列加上第i列 $i=\textcircled{n}, n-1, \cdots 2$

$$|A| = \begin{vmatrix} (1+2+\cdots+n) & (2+3+\cdots+n) & \cdots & n \\ 0 & -1 & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & -(n-1) \end{vmatrix} = \frac{n(n+1)}{2}\cdot(-1)\cdots\cdots-(n-1)$$

$$= \frac{(-1)^{n-1}}{2}\cdot(n+1)!$$

2. 全部加到 1st row: 逐差

$$|A| = \frac{n(n+1)}{2}\cdot\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & 0 \\ n & 1 & & n-2 & n-1 & 1 \\ n-1 & n & & n-3 & n-2 & 1 \\ \vdots & \vdots & & \vdots & \vdots & 1 \\ 2 & 3 & & n & 1 & 1-n \end{vmatrix} = \frac{n(n+1)}{2}\begin{vmatrix} 1 & 0 & \cdots & 0 \\ n & 1-n & 1 & \cdots & 1 \\ n-1 & 1 & 1-n & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \\ 2 & 1 & 1 & \cdots & 1-n \end{vmatrix}$$

求和

n−2.

!横着看.

$$= \begin{vmatrix} -1 & -1 & \cdots & -1 \\ 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-n \end{vmatrix} = \begin{vmatrix} -1 & -1 & \cdots & -1 \\ 0 & -n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -n \end{vmatrix} = (-1)\cdot(-n)^{n-2}$$

$$\frac{n(n+1)}{2}\cdot(-1)^{n-1}\cdot n^{n-2}$$

$$= \boxed{\frac{(-1)^{n-1}}{2}(n+1)\,n^{n-1}}$$ ✓

*若直接逐差:

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ n & 1-n & 1 & \cdots & 1 \\ \vdots & 1 & 1-n & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 2 & 1 & 1 & \cdots & 1-n \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ n-1 & -n & 0 & \cdots & 0 \\ n-2 & 0 & -n & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 0 & \cdots & & -n \end{vmatrix}$$

爪型

3.

(1)

$$\begin{vmatrix} a_1-\sum\limits_{i=2}^{n}\frac{c_i b_i}{a_i} & 0 & \cdots & 0 \\ c_2 & a_2 & & \\ \vdots & & \ddots & \\ c_n & & & a_n \end{vmatrix} = \left(a_1-\sum_{i=2}^{n}\frac{b_i c_i}{a_i}\right)a_2a_3\cdots a_n = \underline{\prod_{i=1}^{n}a_i - \sum_{i=2}^{n}\frac{\left(\prod\limits_{i=2}^{n}a_i\right)b_ic_i}{a_i}}$$ ✓

全减去 1st row:

(2)

$$\begin{vmatrix} a_1 & b_2 & \cdots & b_n \\ b_1-a_1 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ b_n-a_1 & 0 & & a_n \end{vmatrix} = \left(a_1-\sum_{i=2}^{n}\frac{b_i(b_i-a_1)}{a_1}\right)a_2a_3\cdots a_n.$$

4. 1st row + 2nd row :

$$\underset{\substack{\| \\ 0}}{(a+b+c)}\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a+b+c)(\underset{c}{\cancel{a}}-\underset{a}{\cancel{b}})(\underset{c}{\cancel{a}}-\underset{b}{\cancel{c}})(b-\underset{}{\cancel{a}}) = 0$$

5. 右 ~~加~~ 添加一列 →约倒算系数:

$$\begin{vmatrix} \cancel{1} & 1 & \cdots & 1 & 1 \\ \cancel{x} & x_1 & & x_n & x \\ \cancel{x^2} & x_1^2 & & x_n^2 & x^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \cancel{x^{n-2}} & x_1^{n-2} & & x_n^{n-2} & x^{n-2} \\ \boxed{\cancel{x^{n-1}}} & x_1^{n-1} & & x_n^{n-1} & x^{n-1} \\ \cancel{x^n} & x_1^n & & x_n^n & x^n \end{vmatrix}_{(n+1)\times(n+1)} = \prod_{i=1}^{n}(\cancel{x}-x_i)\prod_{1\le j<i\le n}(x_i-x_j) = \cdots + \underbrace{\boxed{A_{n1}}\,x^{n-1}} + \cdots$$

右－左
大－小

$$A_{n1} = \overset{(-1)^{n+(n+1)}}{\cancel{(-1)^{n+1}}}\boxed{M_{n1}}$$ ←标行列式

$$\to = \prod_{1\le j<i\le n}(x_i-x_j)\cdot\sum_{i=1}^{n}(-x_i)\cdot x^{n-1}$$

$$\Rightarrow |D| = M_{n1} = \cancel{(-1)^n}\left(\sum_{i=1}^{n}x_i\right)\left(\prod_{1\le j<i\le n}(x_i-x_j)\right)$$

6. (1) ① 升阶

$$|A| = \begin{vmatrix} 1 & \cancel{1+a_1}b_1 & b_2 & \cdots & b_n \\ 0 & 1+a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ 0 & a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & & & \\ 0 & a_nb_1 & a_nb_2 & \cdots & 1+a_nb_n \end{vmatrix} \begin{matrix} a_1 \\ a_2 \\ a_n \end{matrix} = \begin{vmatrix} 1 & b_1 & b_2 & \cdots & b_n \\ -a_1 & 1 & & & \\ -a_2 & & 1 & & \\ \vdots & & & \ddots & \\ -a_n & & & & 1 \end{vmatrix}$$

recall: $$\underset{\substack{\| \\ 1}}{\boxed{\prod_{i=1}^{n}a_i}} - \sum_{i=2}^{n}\frac{\boxed{\prod_{j=2}^{n}a_j}\,b_ic_i}{\underset{\substack{\| \\ 1}}{a_i}} = 1-\sum_{i=1}^{n}(-a_i)b_i = \underline{1+\sum_{i=1}^{n}a_ib_i}$$ ✓

~~(2)~~ ② $A_{n\times k}$ $B_{k\times n}$ → $(AB)_{n\times n}$, $(BA)_{k\times k}$ $\quad\underbrace{|E_n - AB| = |E_k - BA|}$

if $\underline{n \gg k}$. 降阶.

proof: $$\begin{vmatrix} E_k & B_{k\times n} \\ A_{n\times k} & E_n \end{vmatrix} \overset{\text{2nd row} - A\times\text{1st row}}{=} \begin{vmatrix} E_k & B_{k\times n} \\ 0 & E_n - AB \end{vmatrix} = |E_n - AB|$$

$$\|$$

1st row − B × 2nd row $$\begin{vmatrix} E_k - BA & 0 \\ A_{n\times k} & E_n \end{vmatrix} = |E_k - BA|.$$

$$|A| = \left|E_n \overset{-}{\oplus} \begin{pmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_n \end{pmatrix}(b_1, b_2, \cdots b_n)\right| = \left|\overset{\substack{1 \\ \|}}{E_1} - (b_1, b_2, \cdots b_n)\begin{pmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_n \end{pmatrix}\right| = 1+\sum_{i=1}^{n}a_ib_i$$

(2) $$|A| = \left|\begin{pmatrix} a_1 & 1 \\ a_2 & 1 \\ \vdots & \vdots \\ a_n & 1 \end{pmatrix}\begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ 1 & 1 & \cdots & 1 \end{pmatrix} - E_n\right| = (-1)^n\left|E_n - (\quad)(\quad)\right|$$

$$= (-1)^n\left|E_2 - \begin{pmatrix} b_1 & \cdots & b_n \\ 1 & \cdots & 1 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ \vdots & \vdots \\ a_n & 1 \end{pmatrix}\right|$$

$$\|$$

$$(-1)^n\left|E_2 - \begin{pmatrix} \sum a_ib_i & \sum a_i \\ \sum a_i & n \end{pmatrix}\right| = \cdots\cdots$$

7.

(1) ×100 ×10
$$\begin{vmatrix} 2 & 2 & 1 \\ 2 & 3 & 8 \\ 2 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 2 & 2 & \boxed{221} \\ 2 & 3 & \boxed{238} \\ 2 & 8 & \boxed{289} \end{vmatrix}$$

(2) ×10
$$\begin{vmatrix} 22 & 1 & 2 \\ 23 & 8 & 3 \\ 28 & 9 & 1 \end{vmatrix} = \begin{vmatrix} 22 & \boxed{221} & 2 \\ 23 & \boxed{238} & 3 \\ 28 & \boxed{289} & 1 \end{vmatrix}$$

(3) ×100 ×10

$\Delta$
$$\begin{vmatrix} 2 & 2 & 1 \\ 2 & 3 & 8 \\ 8 & 2 & 9 \end{vmatrix} = \begin{vmatrix} 2 & 2 & 221 \\ 2 & 3 & 238 \\ 8 & 2 & \textcircled{829} \end{vmatrix} \rightarrow 17 \nmid 829$$

(4) ×100 ×10
$$\begin{vmatrix} 1 & 2 & 2 \\ 8 & 2 & 3 \\ 9 & 2 & 8 \end{vmatrix} = \begin{vmatrix} \boxed{221} & \cdots \\ \boxed{238} & \\ \boxed{289} & \end{vmatrix}$$

8.
$$|A| = \sum (\pm)\, \underbrace{a_{1i_1} a_{2i_2} \cdots a_{ni_n}}$$

0个数 > $n^2 - n$ → ∀ n项必有0.

9.
$$\underline{M_{31}} + 4A_{32} + 9M_{33} - 16M_{34} = A_{31} + 4A_{32} + 9A_{33} + 16A_{34}$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 16 & 81 & 256 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 3 & 8 & 15 \\ 0 & 15 & 80 & 255 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ & 1 & 2 & 3 \\ & 0 & 2 & 6 \\ & 0 & 50 & 210 \end{vmatrix}$$

非范德蒙. 用行变换.

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ & 1 & 2 & 3 \\ & & 2 & 6 \\ & & 0 & 60 \end{vmatrix} = \underline{120}$$

+50

$210 - 6 \times 25 = \cancel{6}0$

*
$$\begin{vmatrix} & 1 & 1 & 1 & 1 & 1 \\ x & 1 & 2 & 3 & 4 & x \\ x^2 & 1 & 4 & 9 & 16 & x^2 \\ \textcircled{x^3} & 1 & 8 & 27 & 64 & \textcircled{x^3} \\ x^4 & 1 & 16 & 81 & 256 & x^4 \end{vmatrix}$$

$\binom{5}{2} = 10$项

$$= \underbrace{(x-1)(x-2)(x-3)(x-4)}_{\cdots (-10) \cdot x^3 \cdots} \cdot \left( (-1)(-2)(-3) / (-1)(-2) / (-1) \right)$$

$= 12$

$= -120x^3$

$(-1)^{4+5} \downarrow$

$\underline{120}$.