① 画图.

② 对称性　　曲线积分.

③ 代入

例1. $I=\int_C e^{-(x^2+y^2)}[\cos(2xy)dx+\sin(2xy)dy]$　　$C: x^2+y^2=1 \longrightarrow D: x^2+y^2\le 1$.

★代入 $I=e^{-1}\int_C \cos(2xy)dx+\sin(2xy)dy$, $P=\cos(2xy)$. $Q=\sin(2xy)$. $P_y=-2x\sin(2xy)$, $Q_x=2y\cos(2xy)$.

$=e^{-1}\iint_D \underbrace{2y\cos(2xy)}_{f}+\underbrace{2x\sin(2xy)}_{g}\,dxdy$.　固定$x$, $f(x,-y)=-f(x,y)$.

固定x, $g(x,-y)=-g(x,y)$.

由对称性 $=0$.

例2. I_1: $P=\frac{-y}{x^2+y^2}$, $Q=\frac{x}{x^2+y^2}$, $P_y=\frac{-(x^2+y^2)+y(2y)}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}$, 类似有 $Q_x=\frac{y^2-x^2}{(x^2+y^2)^2}$

由 $Q_x=P_y$.

★代入.

(1). 无奇点. $I_1=\iint_D 0\,dxdy=0$.　(2) 有奇点, $I_1=\int_L xdy-ydx \overset{参数化}{=} \int_0^{2\pi}\cos t\cdot\cos t\,dt+\sin t\cdot\sin t\,dt$

$=\int_0^{2\pi}1\,dt=2\pi$.

I_2: $P_y=Q_x$ (自证) $=\frac{4y^2-x^2}{(x^2+4y^2)^2}$　★挖洞技巧

(1) 无奇点, $I_2=0$　(2) 有奇点, 令 $L_1^+: x^2+4y^2=\varepsilon^2\ (\varepsilon>0)$ 正向.　$\rightarrow \frac{x^2}{\varepsilon^2}+\frac{y^2}{\frac{\varepsilon^2}{4}}=1$

此时, $I_2=\int_{L\cup L_1^-}+\int_{L_1^+}=\iint_D 0\,dxdy+\int_{L_1^+}\frac{xdy-ydx}{x^2+4y^2}=\int_{L_1^+}\frac{1}{\varepsilon^2}(xdy-ydx)$　$\begin{cases}x=\varepsilon\cos t\\ y=\frac{\varepsilon}{2}\sin t\end{cases}$ $t:0\to2\pi$.

$\overset{参数化}{=}\frac{1}{\varepsilon^2}\int_0^{2\pi}\varepsilon\cos t\cdot\frac{\varepsilon}{2}\cos t\,dt+\frac{\varepsilon}{2}\sin t\cdot\varepsilon\sin t\,dt=\frac{1}{\varepsilon^2}\cdot\int_0^{2\pi}\frac{\varepsilon^2}{2}dt=\frac{1}{2}\cdot2\pi=\pi$.

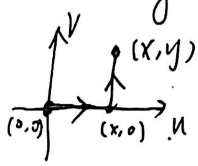
例3: 注意 $\boxed{代入}$ $\boxed{对称性}$. 由 $\frac{x^2}{4}+\frac{y^2}{3}=1 \Rightarrow 3x^2+4y^2=12$ on L.

$\Rightarrow I=\oint_L \underbrace{(2xy)}_{奇\to0}ds+\oint_L 12ds=12$ 周长 $=12a$.

例4: $P=\sin x\sin2y$. $Q=-2\cos x\cos2y$. $P_y=2\sin x\cos2y$, $Q_x=2\sin x\cos2y$, $Q_x=P_y$ (应用题/解答题需先写明该点)

$x\to u$, $y\to v$ 更换积分变量.

$f(x,y)=\int_{(0,0)}^{(x,y)}\sin u\sin2v\,du-2\cos u\cos2v\,dv+f(0,0)$.



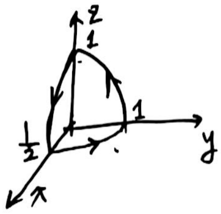
$L_1: v=0$, $u:0\to x$.

$L_2: u=x$, $v:0\to y$.

取直线. $=\int_{L_1}+\int_{L_2}(\ \ )+C=\int_{0\ (L_1)}^{x}0\,du+\int_{0\ (L_2)}^{y}-2\cos x\cos2v\,dv+C$.

$=-\cos x\sin2y+C$.　　$\Rightarrow \cos x\sin2y=C$ 是通解

例5: $z=1-4x^2-y^2$ 抛物曲面.



$$\begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2-\cos z & 2xz^2 & 2xyz+x\sin z \end{vmatrix}.$$

$$= (2xz-4xz)dydz-(2yz+\sin z-2yz-\sin z)dzdx+(2z^2-z^2)dxdy.$$

$$I=\iint_{\Sigma} -2xz\,dydz+z^2dxdy.$$ ① 直接算.

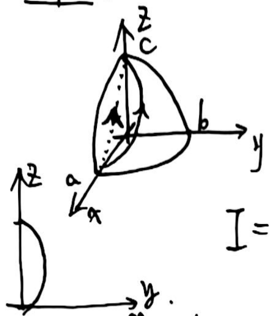
② 补: 左 $\Sigma_1: y=0$

下 $\Sigma_2: z=0$.

后 $\Sigma_3: x=0$.

$$\Rightarrow I=\oiint_{\Sigma\cup\Sigma_1\cup\Sigma_2\cup\Sigma_3}-\iint_{\Sigma_1}-\iint_{\Sigma_2}-\iint_{\Sigma_3}=\iiint_{\Omega}(\underbrace{-2z+2z}_{0})dxdydz-0-0-0=0.$$

*例6: $\boxed{参数化围线}\to\boxed{Stokes}\to\boxed{补封闭}\to\boxed{直线}$.



$$L_1^+: \begin{cases} \frac{x}{a}+\frac{z}{c}=1 \\ y=0. \end{cases} \quad a\to c.$$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}=-1i-1j-1k.$$

$$I=\oint_{L\cup L_1^-}+\int_{L_1^+}=\iint_{\Sigma}-dydz-dzdx-dxdy+\int_{L_1^+}x\,dz$$

$\iint_{\Sigma} dydz=?$ Σ在yoz $(x=0)$: $\begin{cases} \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \\ \frac{x}{a}+\frac{z}{c}=1 \end{cases}$ 消去$x\Rightarrow \begin{cases} \frac{y^2}{b^2}+\frac{2}{c^2}(z-\frac{c}{2})^2=\frac{1}{2} \\ x=0 \end{cases}$ 半椭圆面积为 $\iint_{\Sigma} dydz=\pi\cdot\frac{b}{\sqrt2}\cdot\frac{c}{2}\cdot\frac{1}{2}=\frac{\sqrt2}{8}bc\pi$.

类似有 $\iint_{\Sigma} dxdy=\frac{\sqrt2}{8}ab\pi$. 而 $\iint_{\Sigma} dzdx=0$, 参数化L_1 $z=c-\frac{c}{a}x$ 或 $x=a-\frac{a}{c}z$.

$$\int_{L_1^+}x\,dz=\int_0^c(a-\frac{a}{c}z)dz=a(z-\frac{z^2}{2c})\Big|_0^c=a(c-\frac{c}{2})=\frac{1}{2}ac.$$

$I=-\frac{\sqrt2}{8}bc\pi-\frac{\sqrt2}{8}ab\pi+\frac{1}{2}ac$. (注: $a=b=c=1$时, $I=\frac{1}{2}-\frac{\sqrt2}{4}\pi$).

例7: (1) 验证 $Q_x=P_y$ 略.

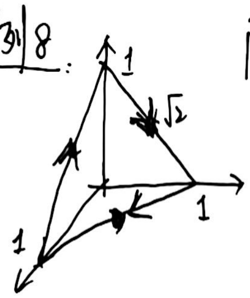
(2) $\boxed{猜}$. 展开. $(\frac{1}{y}+yf(xy))dx+(xf(xy)-\frac{x}{y^2})dy$.

设 $f(x)$ 的一个原函数为 $F(x)$.

猜 $u(x,y)=\frac{x}{y}+F(xy)$. 验证. $u_x=\frac{1}{y}+yf(xy)$, $u_y=-\frac{x}{y^2}+xf(xy)$.

$$\Rightarrow I=\int_{(a,b)}^{(c,d)}(\quad)=u(c,d)-u(a,b)=\frac{c}{d}+F(cd)-\frac{a}{b}-F(ab)=\frac{c}{d}-\frac{a}{b}$$

例8: $\boxed{封闭曲线}\to\boxed{Stokes}$. $\vec n=(-\frac{1}{\sqrt3},-\frac{1}{\sqrt3},-\frac{1}{\sqrt3})$.



直接计算 ✓

$$\begin{vmatrix} -\frac{1}{\sqrt3} & -\frac{1}{\sqrt3} & -\frac{1}{\sqrt3} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}=\sqrt3$$

$$I=\oiint_{T}\sqrt3\,dS=\sqrt3\cdot\frac{\sqrt3}{4}\cdot(\sqrt2)^2=\frac{3}{2}.$$