同济大学
TONGJI UNIVERSITY
地址：中国上海市四平路1239号　邮编：200092
1239 SIPING ROAD SHANGHAI CHINA 200092
电话（TEL）：+86 21-　传真（FAX）：+86 21-
网址（WEB）：www.tongji.edu.cn

11. $\frac{\partial g}{\partial x}=y+\frac{yf_1'}{x^2}-\frac{f_2'}{y}$　　$\frac{\partial g}{\partial y}=x-\frac{f_1'}{x}+\frac{xf_2'}{y^2}$

$\frac{\partial^2 g}{\partial x^2}=y\cdot\frac{[f_{11}''(-\frac{y}{x^2})+f_{12}''\frac{1}{y}]x^2-f_1'\cdot 2x}{x^4}-\frac{1}{y}[f_{21}''(-\frac{y}{x^2})+f_{22}''\frac{1}{y}]$

$\frac{\partial^2 g}{\partial x\partial y}=1+\frac{1}{x^2}[f_1'+y(f_{11}''\frac{1}{x}+f_{12}''(-\frac{x}{y^2}))]-\frac{1}{y^2}\{[f_{21}''\frac{1}{x}+f_{22}''\cdot(-\frac{x}{y^2})]y-f_2'\}$

$\frac{\partial^2 g}{\partial y^2}=-\frac{1}{x}[f_{11}''\frac{1}{x}+f_{12}''(-\frac{x}{y^2})]+x\cdot\frac{[f_{21}''\frac{1}{x}+f_{22}''(-\frac{x}{y^2})]y^2-2yf_2'}{y^4}$

$\therefore x^2\frac{\partial^2 g}{\partial x^2}+2xy\frac{\partial^2 g}{\partial x\partial y}+y^2\frac{\partial^2 g}{\partial y^2}=2xy$　$\square$

12. 面积为 $S=\pi ab$，即求 ab 最小值

设切点为 (x_0,y_0)，$x_0\geq 0, y_0\geq 0$，则 $\begin{cases}(x_0-1)^2+y_0^2=1\\ \frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1\end{cases}$

得 $(\frac{b^2}{a^2}-1)x_0^2+2x_0-b^2=0$

$a=b$ 时，$x_0=\frac{b^2}{2}$，此时只在 $(2,0)$ 处相切　$\therefore b^2=4$　即 $a=b=2$，$S=4\pi$

$a\neq b$ 时，方程只有两个相等的解　$\therefore \Delta=4+4b^2(\frac{b^2}{a^2}-1)=0$　即 $b^4-a^2b^2+a^2=0$

$\therefore$ 令 $F(a,b,\lambda)=\pi ab+\lambda(b^4-a^2b^2+a^2)$　$\therefore \begin{cases}F_a'=\pi b+\lambda(-2ab^2+2a)=0\\ F_b'=\pi a+\lambda(4b^3-2a^2b)=0\\ F_\lambda'=b^4-a^2b^2+a^2=0\end{cases}$

$\therefore a=\frac{3\sqrt{2}}{2}, b=\frac{\sqrt{6}}{2}, S=\frac{3\sqrt{3}}{2}\pi$　$\therefore a=\frac{3\sqrt{2}}{2}, b=\frac{\sqrt{6}}{2}$ 时面积最小　$\square$

13. 即求 $\sqrt{x^2+y^2}$ 的最值

令 $F(x,y,\lambda)=x^2+y^2+\lambda(x^2+xy+y^2+2x-2y-12)$　$\therefore \begin{cases}F_x'=2x+\lambda(2x+y+2)=0\\ F_y'=2y+\lambda(x+2y-2)=0\\ F_\lambda'=x^2+xy+y^2+2x-2y-12=0\end{cases}$

$\therefore$ 解得 $\begin{cases}x=-6\\ y=6\end{cases}$，$\begin{cases}x=2\\ y=-2\end{cases}$，$\begin{cases}x=1+\frac{\sqrt{21}}{5}\\ y=-1+\frac{\sqrt{21}}{5}\end{cases}$，$\begin{cases}x=1-\frac{\sqrt{21}}{3}\\ y=-1-\frac{\sqrt{21}}{3}\end{cases}$

依次代入 $\sqrt{x^2+y^2}$，得 $d_{max}=6\sqrt{2}$，$d_{min}=\frac{2\sqrt{15}}{5}$

14. 用极坐标换元. $\iint\limits_D \frac{1}{3x^2+y^2}dxdy = \int_0^{\frac{\pi}{3}} d\theta \int_{\sqrt{\frac{1}{1-\sin\theta\cos\theta}}}^{\sqrt{\frac{2}{1-\sin\theta\cos\theta}}} \frac{1}{r^2(3\cos^2\theta+\sin^2\theta)}\cdot r\,dr$

$= \frac{\ln 2}{2}\int_0^{\frac{\pi}{3}} \frac{1}{3\cos^2\theta+\sin^2\theta}d\theta = \frac{\ln 2}{2}\int_0^{\frac{\pi}{3}}\frac{1}{3+\tan^2\theta}d(\tan\theta) = \frac{\ln 2}{2}\left(\frac{1}{\sqrt3}\arctan\frac{\tan\theta}{\sqrt3}\right)\Big|_0^{\frac{\pi}{3}} = \frac{\sqrt3\ln 2}{24}\pi$ □

15. 用直角坐标表示 $\iint\limits_D ye^{\frac{y}{x}}dxdy = \int_1^2 dx\int_{\frac{\sqrt3}{3}x}^{\sqrt3 x} ye^{\frac{y}{x}}dy$

令 $\begin{cases} u=\frac{y}{x} \\ v=x \end{cases}$ ∴ $D'=\{(u,v)\mid \frac{\sqrt3}{3}\le u\le\sqrt3,\ 1\le v\le 2\}$ $\begin{cases} x=v \\ y=uv \end{cases}$

∴ $|J| = \begin{vmatrix} 0 & 1 \\ v & u \end{vmatrix} = v$ ∴ $\int_1^2 dx\int_{\frac{\sqrt3}{3}x}^{\sqrt3 x} ye^{\frac{y}{x}}dy = \int_1^2 v^2dv\int_{\frac{\sqrt3}{3}}^{\sqrt3} ue^u du$

$= \frac{7}{3}(u-1)e^u\Big|_{\frac{\sqrt3}{3}}^{\sqrt3} = \frac{7}{3}(\sqrt3-1)\left(e^{\sqrt3}+\frac{\sqrt3}{3}e^{\frac{\sqrt3}{3}}\right)$ □

16. D关于 $x=\pi$ 对称

∴ $\iint\limits_D(x+2y)dxdy = \iint\limits_D(x-\pi)dxdy + \iint\limits_D(\pi+2y)dxdy$ $\quad \iint\limits_D(x-\pi)dxdy=0$

$= \iint\limits_D(\pi+2y)dxdy$

$= \int_0^{2\pi}dx\int_0^{y(x)}(\pi+2y)dy = \int_0^{2\pi} \pi y(x)+y^2(x)\,dx$ 将 $\begin{cases} x=t-\sin t \\ y=1-\cos t \end{cases}$ 代入

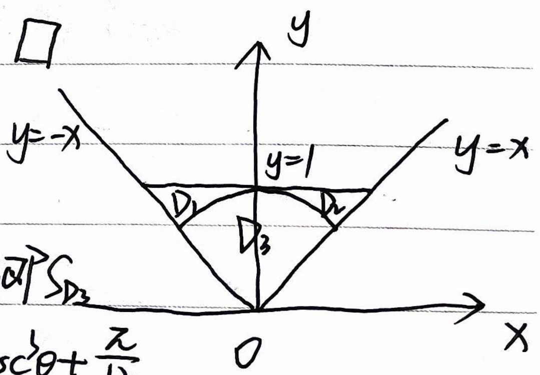
$= \int_0^{2\pi}[\pi(1-\cos t)+(1-\cos t)^2](1-\cos t)dt = \pi\int_0^{2\pi}(1-\cos t)^2dt + \int_0^{2\pi}(1-\cos t)^3dt$

$= \pi\int_0^{2\pi}4\sin^4\frac{t}{2}dt + \int_0^{2\pi}8\sin^6\frac{t}{2}dt$, 令 $\frac{t}{2}=\theta$

则上式 $= 8\pi\int_0^{\pi}\sin^4\theta\, d\theta + 16\int_0^{\pi}\sin^6\theta\, d\theta$ (Wallis公式)

$= 8\pi\cdot 2\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + 16\cdot 2\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \pi(3\pi+5)$ □

17. $f(x,y) = \begin{cases} \sqrt{x^2+y^2}, & (x,y)\in D_1\cup D_2 \\ 1, & (x,y)\in D_3 \end{cases}$



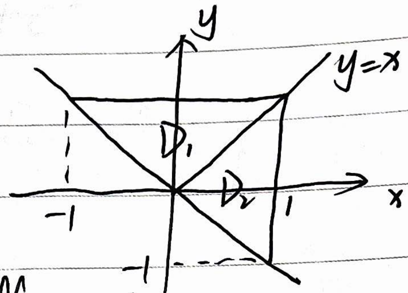
∴ $\iint\limits_D f(x,y)d\sigma = \iint\limits_{D_1\cup D_2}\sqrt{x^2+y^2}dxdy + \iint\limits_{D_3}dxdy$ 前者用极坐标,后者用 S_{D_3}

$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}d\theta\int_1^{\frac{1}{\sin\theta}} r\cdot r\,dr + \frac{\pi}{4} = \frac{1}{3}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}(\csc^3\theta-1)d\theta + \frac{\pi}{4} = \frac{1}{3}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\csc^3\theta + \frac{\pi}{12}$

而 $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\csc^3\theta d\theta = -\csc\theta\cot\theta\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} - \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\cot^2\theta\csc\theta d\theta = 2\sqrt2 - \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}(\csc^2\theta-1)\csc\theta d\theta$

$\therefore 2\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \csc^3\theta\, d\theta = 2\sqrt{2} + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \csc\theta\, d\theta = 2\sqrt{2} + \ln|\csc\theta - \cot\theta| \Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = 2\sqrt{2} + 2\ln(\sqrt{2}+1)$

$\therefore \frac{1}{3}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \csc^3\theta\, d\theta + \frac{\pi}{12} = \frac{\pi}{12} + \frac{\sqrt{2}+\ln(\sqrt{2}+1)}{3}$ □



18. 用 $y=x$ 将 D 划为 D_1 与 D_2

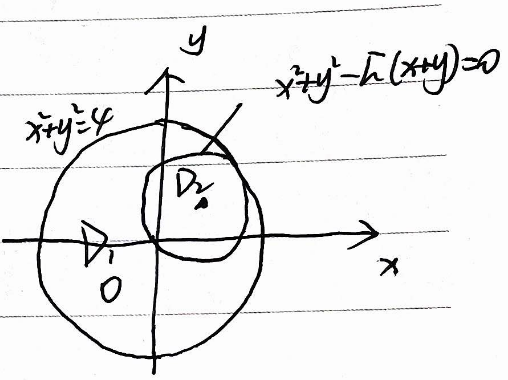
令 $f(x,y) = x+y+(e^x\cos x - e^y\cos y)\sin(xy)$

D 关于 $y=x$ 对称

$\therefore$ 由轮换对称性，$\iint_D f(x,y)\,d\sigma = \frac{1}{2}\iint_D (f(x,y)+f(y,x))\,d\sigma = \iint_D (x+y)\,d\sigma$

再由对称性，$\iint_D (x+y)\,d\sigma = 2\iint_{D_1}(x+y)\,d\sigma$ 又 D_1 关于 y 轴对称，

$= 2\iint_{D_1} y\,d\sigma = 2\int_0^1 y\,dy\int_{-y}^{y} dx = \frac{4}{3}$ □



19. 令 $f(x,y) = x^2+y^2-\sqrt{2}(x+y)$，则 $f(x,y)=0$ 将 D 划为 D_1 和 D_2

$\therefore \iint_D |f(x,y)|\,dxdy = \iint_{D_1} f(x,y)\,dxdy - \iint_{D_2} f(x,y)\,dxdy$

$= \iint_{D_1+D_2=D} f(x,y)\,dxdy - 2\iint_{D_2} f(x,y)\,dxdy$ （补形思想）

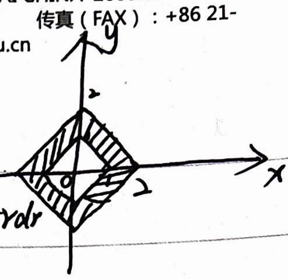
D 关于 x 轴，y 轴对称，则 $\iint_D \sqrt{2}(x+y)\,dxdy = 0$

$\therefore \iint_D f(x,y)\,dxdy = \iint_D x^2+y^2\,dxdy = 4\int_0^{\frac{\pi}{2}} d\theta\int_0^2 r^3\,dr = 8\pi$

$-2\iint_{D_2} f(x,y)\,dxdy \xlongequal{\text{极坐标}} -2\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta\int_0^{\sqrt{2}(\cos\theta+\sin\theta)} [r^2-\sqrt{2}r(\sin\theta+\cos\theta)]\,r\,dr = \frac{2}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} (\sin\theta+\cos\theta)^4\,d\theta$

$= \frac{8}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin^4(\theta+\frac{\pi}{4})\,d\theta \xlongequal{t=\theta+\frac{\pi}{4}} \frac{8}{3}\int_0^{\pi} \sin^4 t\,dt = \frac{8}{3}\cdot 2\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \pi$

$\therefore \iint_D |f(x,y)|\,dxdy = 8\pi+\pi = 9\pi$ □

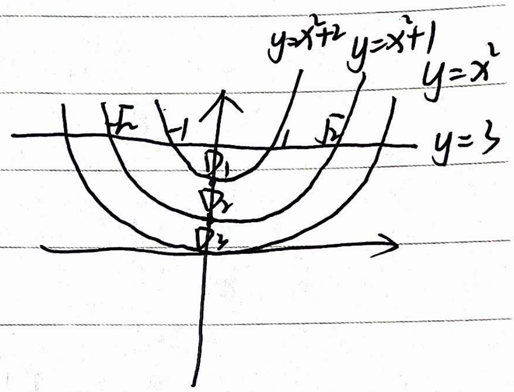


20. 积分区域D为阴影部分，设第一象限阴影区域为D_1，则由对称性，

$$\iint_D \frac{x^2+y^2}{|x|+|y|}dxdy = 4\iint_{D_1}\frac{x^2+y^2}{x+y}dxdy \xlongequal{极坐标} 4\int_0^{\frac{\pi}{2}}d\theta\int_{\frac{1}{\sin\theta+\cos\theta}}^{\frac{2}{\sin\theta+\cos\theta}}\frac{r}{\sin\theta+\cos\theta}\,r\,dr$$

$$=\frac{28}{3}\int_0^{\frac{\pi}{2}}\frac{1}{(\sin\theta+\cos\theta)^4}d\theta=\frac{7}{3}\int_0^{\frac{\pi}{2}}\csc^4(\theta+\tfrac{\pi}{4})d\theta=-\frac{7}{3}\int_0^{\frac{\pi}{2}}\left[\cot^2(\theta+\tfrac{\pi}{4})+1\right]d\left[\cot(\theta+\tfrac{\pi}{4})\right]$$

$$=-\frac{7}{3}\left[\frac{1}{3}\cot^3(\theta+\tfrac{\pi}{4})+\cot(\theta+\tfrac{\pi}{4})\right]\Bigg|_0^{\frac{\pi}{2}}=\frac{56}{9}\quad\square$$



21. 由$y=x^2+2$，$y=x^2+1$将D划为D_1, D_2, D_3

$$\therefore \sqrt{[y-x^2]}=\begin{cases}0, & (x,y)\in D_3\\ 1, & (x,y)\in D_2\\ \sqrt{2}, & (x,y)\in D_1\end{cases}$$

$$\therefore \iint_D\sqrt{[y-x^2]}dxdy=\iint_{D_1}\sqrt{2}dxdy+\iint_{D_2}dxdy=\sqrt{2}S_{D_1}+S_{D_2}=(\sqrt{2}-1)S_{D_1}+(S_{D_1}+S_{D_2})$$

$$=(\sqrt{2}-1)\int_{-1}^{1}(3-x^2-2)dx+\int_{-\sqrt{2}}^{\sqrt{2}}(3-x^2-1)dx=4\sqrt{2}-\frac{4}{3}\quad\square$$

22. 可直接积分，下面用换元法：

令$\begin{cases}u=x+y\\ v=\frac{y}{x}\end{cases}$ $\therefore 1\le u\le 2,\ 1\le v\le 2$ $\begin{cases}x=\frac{u}{1+v}\\ y=\frac{uv}{1+v}\end{cases}$ $\therefore |J|=\begin{vmatrix}\frac{1}{1+v} & -\frac{u}{(1+v)^2}\\ \frac{v}{1+v} & \frac{u}{(1+v)^2}\end{vmatrix}=\frac{u}{(1+v)^2}$

$$\therefore \iint_D(x+y)dxdy=\int_1^2u^2du\int_1^2\frac{1}{(1+v)^2}dv=\frac{7}{3}\cdot\frac{1}{6}=\frac{7}{18}\quad\square$$