定积分

——我不是why 2024.12.

一.

(1) 原式 $= \frac{1}{2\pi} \cdot \int_0^{\pi} \sin x \, dx = \frac{1}{\pi}$.

(2) 原式 $= \int_0^1 \frac{1}{x+1} dx = \ln 2$.

(3) 原式 $= \frac{\int_0^1 x^{\alpha} dx}{\int_0^1 x^{\alpha+1} dx} = \frac{\frac{1}{\alpha+1}}{\frac{1}{\alpha+2}} = \frac{\alpha+2}{\alpha+1}$.

(4) 注意到: $\frac{1}{n^2+(k+1)^2} \leq \frac{1}{n^2+k^2+1} \leq \frac{k^2}{n^2}$

从而:

$$\sum_{k=1}^{n} \frac{n}{n^2+(k+1)^2} \leq \sum_{k=1}^{n} \frac{n}{n^2+k^2+1} \leq \sum_{k=1}^{n} \frac{n}{n^2+k^2}$$

$$= \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+\left(\frac{k+1}{n}\right)^2} \qquad = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+\left(\frac{k}{n}\right)^2}$$

令 $n \to \infty$,

$$= \int_0^1 \frac{1}{x^2+1} dx \qquad = \int_0^1 \frac{1}{x^2+1} dx$$

夹逼得 $\pi/4$.

(5) $\sum_{k=1}^{n} \sin\frac{k}{n} \sin\frac{k}{n^2} = \sum_{k=1}^{n} \sin\frac{k}{n} \left(\frac{k}{n^2} - \frac{k^3}{6n^6} + o\left(\frac{1}{n^3}\right)\right)$

$$= \sum_{k=1}^{n} \sin\frac{k}{n} \frac{k}{n^2} - o\left(\frac{1}{n^2}\right)$$

令 $n \to \infty$.

原式 $= \int_0^1 x \sin x \, dx$ 是多少呢?

二.

(1) $\int_0^1 \arcsin x\,dx = x\arcsin x\Big|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}dx$

$= 1\arcsin 1 + \frac{1}{2}\int_0^1 \frac{d(1-x^2)}{\sqrt{1-x^2}}$

$= 1\arcsin 1 + \frac{1}{2}\times 2(1-x^2)^{\frac{1}{2}}\Big|_0^1 = 1\arcsin 1 - 1$

$= \frac{\pi}{2} - 1.$

(2) $\int_1^e \sin(\ln x)\,dx \xlongequal[\substack{x=e^t \\ dx=e^t dt}]{\ln x = t} \underbrace{\int_0^1 \sin t\, e^t\,dt}$ (利用组合积分法)

考查 $\int_0^1 \cos t\, e^t\,dt$.

(3) $\int_1^{\sqrt{2}} \frac{dx}{x\sqrt{1+x^2}} = \int_1^{\sqrt{2}} \frac{\frac{1}{x^2}dx}{\sqrt{1+(\frac{1}{x})^2}} = -\int_1^{\sqrt{2}} \frac{d(\frac{1}{x})}{\sqrt{1+(\frac{1}{x})^2}} = -\ln\left(1+\sqrt{1+\frac{1}{x}}\right)\Bigg|_1^{\sqrt{2}}$.

(4) 对称后用点火公式. 原式 $= \begin{cases} 0. & n奇 \\ \frac{(n-1)!!}{n!!}\pi. & n偶. \end{cases}$

(5) $\int_1^e x\ln^n x\,dx = \frac{1}{2}x^2\ln^n x\Big|_1^e - \frac{n}{2}\int_1^e x\ln^{n-1}dx$

$I_n = \frac{1}{2}e^2 - \frac{n}{2}I_{n-1}$ (每次分部都会出来一个$\frac{e^2}{2}$)

(6) $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}dx = -\frac{\pi}{2}\int_0^\pi \frac{d\cos x}{1+\cos^2 x}$.

$= -\frac{\pi}{2}\arctan\cos x\Big|_0^\pi$

(7) $\int_{-\pi}^{\pi} \frac{6x(x^2+\sin x)}{1+\cos^2 x}dx \xlongequal[偶!]{奇!} 2\int_0^\pi \frac{6x\sin x}{1+\cos^2 x}dx \xrightarrow{(同(6))}$

(8) $\int_0^{8\pi} x|\sin x|dx = \int_0^{\pi} x\sin x dx - \int_{\pi}^{2\pi} x\sin x dx$

$+ \cdots - \int_{7\pi}^{8\pi} x\sin x dx$

事实上, $\int_{(k-1)\pi}^{k\pi} x\sin x dx = -\int_{(k-1)\pi}^{k\pi} x d\cos x = x\cos x\Big|_{(k-1)\pi}^{k\pi} + \underbrace{\int_{(k-1)\pi}^{k\pi}\cos x dx}_{=0}$

求和即可 ✓

(9) $\int_0^1 \frac{\ln(1+x)}{1+x^2}dx \xlongequal{x=\tan t} \int_0^{\frac{\pi}{4}} \ln(1+\tan t)dt$

利用对称性: $\int_0^{\frac{\pi}{4}} \ln(1+\tan t)dt \xlongequal{\frac{\pi}{4}-t=u} \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-u\right)\right)du = I$

$2I = \int_0^{\frac{\pi}{4}} \ln(1+\tan t) + \ln\left(\frac{2}{1+\tan t}\right) dt = \frac{\pi}{4}\ln 2 \Rightarrow I = \frac{\pi}{8}\ln 2.$

(10) $\int_0^{\frac{\pi}{2}} \frac{f(\sin x)}{f(\sin x)+f(\cos x)}dx \xlongequal{\frac{\pi}{2}-x=u} \int_0^{\frac{\pi}{2}} \frac{f(\cos x)}{f(\sin x)+f(\cos x)}dx$ $\quad I = \frac{\pi}{4}$

$= I$ (想一下!)

三. (1)

$$\lim_{x\to 0}\frac{x^2}{\int_{\cos x}^{1} e^{-u^2}du} = \lim_{x\to 0}\frac{x^2}{\sin x\, e^{-x^2}}$$

(2) $$\lim_{x\to 1}\frac{\int_1^x\left[\int_t^1 (t-u)f(u)du\right]dt}{(x-1)^3} = \lim_{x\to 1}\frac{\int_x^1 (x-u)f(u)du}{3(x-1)^2}$$

$$= \lim_{x\to 1}\frac{\int_x^1 f(u)du - xf(x) + xf(x)}{6(x-1)} = -\frac{1}{6}f(1)$$

(3) $\lim\limits_{x\to 0}\dfrac{\int_0^x t f(x-t)\,dt}{\int_0^x x f(x-t)\,dt} \xlongequal{x-t=u} \lim\limits_{x\to 0}\dfrac{\int_0^x (x-u) f(u)\,du}{\int_0^x x f(u)\,du}$

L-Hospital即可

四. 两边关于1到e积分后显然.

五. (1) $\int_0^\pi x f(\sin x)\,dx \xlongequal{\pi-x=t} \int_0^\pi (\pi-t) f(\sin t)\,dt$ 即得.

(2) $\int_1^4 f\left(\frac{x}{2}+\frac{2}{x}\right)\frac{\ln x}{x}\,dx \underset{t=\frac{4}{x}}{\xlongequal{x=\frac{4}{t}}} \int_4^1 f\left(\frac{t}{2}+\frac{2}{t}\right)\frac{\ln\frac{4}{t}}{\frac{4}{t}}\cdot\frac{-4}{t^2}\,dt$

$dx=\frac{-4}{t^2}\,dt \qquad =\int_1^4 f\left(\frac{t}{2}+\frac{2}{t}\right)\frac{2\ln 2-\ln t}{t}\,dt$

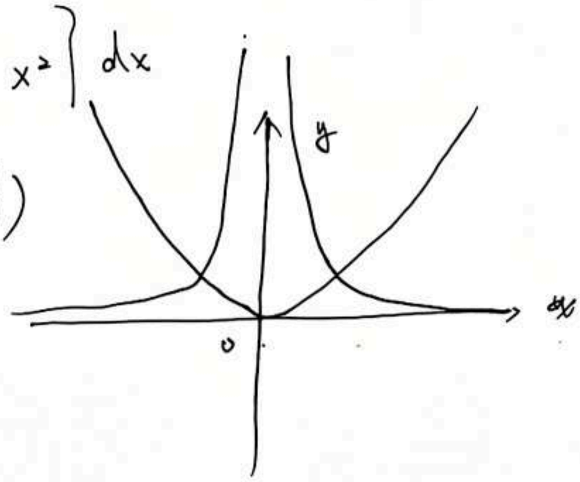
移项除以2即得.

六. (1) $\int_0^n (x-[x])\,dx = \sum\limits_{k=1}^{n}\int_{k-1}^{k}(x-[x])\,dx$

事实上. $\int_{k-1}^{k}(x-[x])\,dx = \int_0^1 (x-[x])\,dx = \frac{1}{2}$. 原式 $= n/2$.

(2) $\int_{-2}^{2} \min\left\{\frac{1}{|x|}, x^2\right\} dx$

很容易注意到 (作图!)



由奇偶性: 原式

$= 2\int_0^2 \min\left\{\frac{1}{|x|}, x^2\right\} dx$

$= 2\left(\int_0^1 x^2\,dx + \int_1^2 \frac{1}{x}\,dx\right)$

(3) $\int_0^2 |1-x|dx = \underline{\int_0^1 (1-x)dx + \int_1^2 (x-1)dx}$

6. $\int_0^1 f(x)dx = \int_0^1 f(x)\,d(x-1) = (x-1)f(x)\Big|_0^1 - \int_0^1 (x-1)f'(x)dx$

$$= -\int_0^1 (x-1)\arctan(x-1)^2 dx = -\frac{1}{2}\int_0^1 \arctan\left[(x-1)^2\right] d(x-1)^2$$

$$= -\frac{1}{2}\int_1^0 \arctan t\,dt$$

$$= \underline{\frac{1}{2}\int_0^1 \arctan t\,dt} \ (分部!)$$

7. (1) 组合积分即可

(2) $\int_2^{+\infty} \frac{dx}{x\ln^p x} = \int_2^{+\infty} \frac{d\ln x}{\ln^p x} = \int_{\ln 2}^{+\infty} \frac{dt}{t^p}$.

(3) $\int_0^{\frac{\pi}{2}} \ln\sin x\,dx \overset{x=2t}{=} \frac{\pi}{2}\ln 2 + \int_0^{\frac{\pi}{4}} 2\ln\sin t\,dt + \underline{\int_0^{\frac{\pi}{4}} 2\ln\cos t\,dt}$

$$\overset{u=\frac{\pi}{2}-t}{=\!=} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\ln\sin t\,dt$$

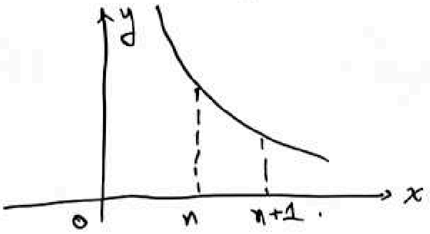
$$\Rightarrow -\frac{\pi}{2}\ln 2 = \int_0^{\frac{\pi}{2}} \ln\sin x\,dx$$

(4) $\int_0^{\frac{\pi}{2}} \frac{\ln(1+\tan x)}{(\cos x+\sin x)^2}dx = \int_0^{\frac{\pi}{2}} \frac{\ln(1+\tan x)\sec^2 x}{(1+\tan x)^2}d\,x$

$$= \int_0^{+\infty} \frac{\ln(1+x)}{(1+x)^2}dx$$

(分部!)

九. 什么是面积原理? $f(x)\geq 0$, 且 $f(x)\downarrow$



满足: $f(n+1)\leq \int_n^{n+1} f(x)dx \leq f(n)$

(1) 代入 $f(x)=\frac{1}{x}$.

$$\frac{1}{n+1}\leq \ln\left(1+\frac{1}{n}\right)\leq \frac{1}{n}\leq \ln\left(1+\frac{1}{n-1}\right)$$

狠狠地求和!

(2) $\sum_{j=1}^{n^2} \frac{n}{n^2+j^2}$ (这是个反常积分! 不能直接用求和化积分.)

固定n! $\Rightarrow \frac{n}{n^2+j^2} = f(x)=\frac{n}{n^2+x^2}\downarrow$ (x增大)

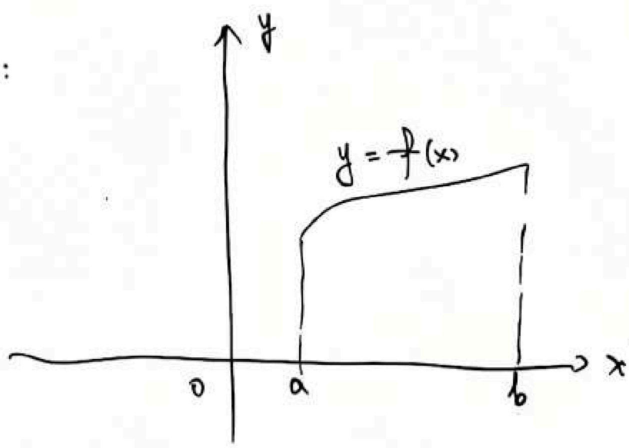
~~且 $\int_{j-1}^{j}\frac{n}{n^2+j^2}dj$~~

$$\int_{j-1}^{j}\frac{n}{n^2+x^2}dx = \arctan\frac{x}{n}\Big|_{j-1}^{j}$$

$$\Rightarrow \frac{n}{n^2+(j-1)^2}\leq \arctan\frac{j}{n}-\arctan\frac{j-1}{n}\leq \frac{n}{n^2+j^2}$$

狠狠地求和!

十. 作图:



$$V=\lim_{n\to\infty}\sum_{i=1}^{n} 2\pi\left(a+\frac{i}{n}(b-a)\right)\frac{b-a}{n} f\left(a+\frac{i}{n}(b-a)\right)$$

$$=2\pi\int_a^b x f(x)dx$$

十一. (1) $g(x)=\int_{-a}^{x}(x-t)f(t)\,dt-\int_{x}^{a}(x-t)f(t)\,dt$

$$=x\int_{-a}^{x}f(t)\,dt-\int_{-a}^{x}tf(t)\,dt+x\int_{a}^{x}f(t)\,dt-\int_{a}^{x}tf(t)\,dt$$

$g'(x)=\int_{-a}^{x}f(t)\,dt+\int_{a}^{x}f(t)\,dt \qquad g''(x)=2f(x)>0.$

$g'(x)\uparrow$ on $[-a,a]$.

(2) $x=0$. 取最小 (高中导数)

(3) $g(0)=-\int_{-a}^{0}tf(t)\,dt-\int_{a}^{0}tf(t)\,dt=2\int_{0}^{a}tf(t)\,dt$

$\boxed{\text{后解}} \quad =f(a)-a^2-1$

这是一个关于a的微分方程. 求解得:

$f(a)+1=Ce^{a^2} \qquad f(0)=1. \Rightarrow C=2.$

$f(x)=2e^{x^2}-1$.

$$\left(\frac{2}{n}=\int_{\frac{k-1}{n}\pi}^{\frac{k}{n}\pi}|\sin nx|\,dx\right)$$

十二. (1) $\ln\left(1+\frac{k-1}{n}\pi\right)\frac{2}{n} \leq \int_{\frac{k-1}{n}\pi}^{\frac{k\pi}{n}}|\sin nx|\ln(1+x)\,dx \leq \ln\left(1+\frac{k}{n}\pi\right)\frac{2}{n}$

(2) 两边求和:

$$\sum_{k=1}^{n}\ln\left(1+\frac{k-1}{n}\pi\right)\frac{2}{n} \leq \int_{0}^{\pi}|\sin nx|\ln(1+x)\,dx \leq \sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\pi\right)\frac{2}{n}$$

$n\to\infty$. $\quad \frac{2}{\pi}\int_{0}^{\pi}\ln(1+x)\,dx \leq$ 这个 $\leq \frac{2}{\pi}\int_{0}^{\pi}\ln(1+x)\,dx$

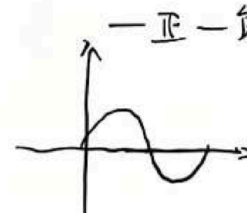
十三. $\int_0^{2\pi} f(x)\sin nx\,dx$

$$=\sum_{k=1}^{n}\int_{\frac{2}{n}(k-1)\pi}^{\frac{2}{n}k\pi} f(x)\sin nx\,dx$$

1不是从 $\sin nx$ 的特点出发. $\int_{\frac{2}{n}(k-1)\pi}^{\frac{2}{n}k\pi} f(x)\sin nx\,dx$

$$\xlongequal[dx=\frac{1}{n}dt]{nx=t}\underbrace{\frac{1}{n}\int_{2(k-1)\pi}^{2k\pi} f\left(\frac{t}{n}\right)\sin t\,dt}_{\text{跨度为 }2\pi}\quad\cdots(*)$$

$\sin x$ 在 $[0,2\pi]$ 上 一正一负



其中: $(*)=\frac{1}{n}\left[\int_{(2k-2)\pi}^{(2k-1)\pi} f\left(\frac{t}{n}\right)\sin t\,dt+\underbrace{\int_{(2k-1)\pi}^{2k\pi} f\left(\frac{t}{n}\right)\sin t\,dt}\right]$

$$\xlongequal{t-\pi=u}\int_{(2k-2)\pi}^{(2k-1)\pi} f\left(\frac{u+\pi}{n}\right)(-\sin u)\,du$$

$$(*)=\frac{1}{n}\int_{(2k-2)\pi}^{(2k-1)\pi}\underbrace{\left[f\left(\frac{t}{n}\right)-f\left(\frac{t+\pi}{n}\right)\right]}_{\geq 0}\underbrace{\sin t}_{\geq 0}\,dt$$

每一项 ≥ 0.

求和 ≥ 0 ✓

十四. 法一: 积分第一中值定理:

f 连续, g 连续且不变号 $\exists\,\xi\in[a,b]$ $\int_a^b f(x)g(x)\,dx=f(\xi)\int_a^b g(x)\,dx$

$$\int_0^{2\pi} f(x)|\sin nx|\,dx=\sum_{k=1}^{n}\int_{\frac{2}{n}(k-1)\pi}^{\frac{2}{n}k\pi} f(x)|\sin nx|\,dx$$

$$=\sum_{k=1}^{n}\frac{1}{n}\int_{2(k-1)\pi}^{2k\pi} f\left(\frac{t}{n}\right)|\sin t|\,dt,$$

满足积分第一中值定理条件! $\xi_k\in\left[\frac{2(k-1)\pi}{n},\frac{2k\pi}{n}\right]$

$$=\sum_{k=1}^{n}\frac{1}{n}f(\xi_k)\underbrace{\int_{(2k-1)\pi}^{2k\pi}|\sin t|\,dt}=4=\sum_{k=1}^{n}\frac{4}{n}f(\xi_k)$$

$$=\frac{4}{2\pi}\int_0^{2\pi} f(x)\,dx.$$

法二：由十二题得到的启发！

As $f(x)$ 是闭区间上的连续函数，在 $\left[\frac{2(k-1)\pi}{n}, \frac{2k\pi}{n}\right]$ 上存在最大(小)值点

分别记为 M_k, m_k.

从而：$(*)$ $f(m_k) \le \int_{\frac{2(k-1)\pi}{n}}^{\frac{2}{n}k\pi} f(x)|\sin nx|\,dx \le \left[\int_{\frac{2(k-1)\pi}{n}}^{\frac{2}{n}k\pi} |\sin nx|\,dx\right] f(M_k)$

$\Downarrow$

$= (*) = \frac{4}{n}$,

再求和夹逼即可！

十五：Taylor展开

(1) 将 $f(x)$ 在 $x=\frac{a}{2}$ 处展开：

$$f(x) = f\left(\frac{a}{2}\right) + f'\left(\frac{a}{2}\right)\left(x-\frac{a}{2}\right) + \frac{f''(\xi)}{2!}\left(x-\frac{a}{2}\right)^2$$

再由 $\int_0^a f(x)\,dx = af\left(\frac{a}{2}\right) + \underbrace{\int_0^a f'\left(\frac{a}{2}\right)\left(x-\frac{a}{2}\right)dx}_{=0} + \underbrace{\int_0^a \frac{f''(\xi)}{2!}\left(x-\frac{a}{2}\right)^2 dx}_{\ge 0}$

$\ge af\left(\frac{a}{2}\right)$！(太成立了)

~~(2) 因凸性 + 基本不等式：~~

$\cancel{\int_0^a f(x)\,dx \ge f\left(\int_0^a x^2\,dx\right) = f(}$

(2) 将 $f(x)$ 在 $x=\frac{a}{3}$ 展开

$$f(x) = f\left(\frac{a}{3}\right) + f'\left(\frac{a}{3}\right)\left(x-\frac{a}{3}\right) + \frac{f''(\xi)}{2!}\left(x-\frac{a}{3}\right)^2$$

$$\ge f\left(\frac{a}{3}\right) + f'\left(\frac{a}{3}\right)\left(x-\frac{a}{3}\right)$$

令 $x=t^2$，再从 0 到 a 积分：$\int_0^a f(t^2)\,dt \ge af\left(\frac{a}{3}\right) + \underbrace{\int_0^a f'\left(\frac{a}{3}\right)\left(t^2-\frac{a}{3}\right)dt}_{=0}$.

$\square$.