例12 设 $\int_0^1 x^2 f(x)dx = A$.

$$A = \int_0^1 x^2\sqrt{1-x^2}\,dx + A\int_0^1 \frac{x^2}{1+x^2}dx$$

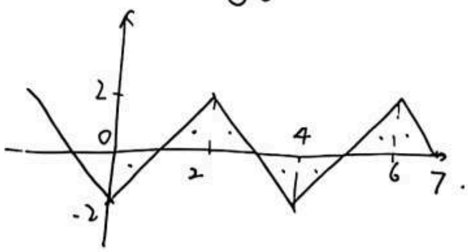
$$A = \int_0^{\frac{\pi}{2}} \sin^2 t\cdot\cos^2 t\,dt + A\cdot\left(x-\arctan x\right)\Big|_0^1$$

$$A = \int_0^{\frac{\pi}{2}} \sin^2 t - \sin^4 t\,dt + A\left(1-\frac{\pi}{4}\right).$$

$$\frac{\pi}{4}A = \frac{1}{2}\cdot\frac{\pi}{2} - \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \qquad \Rightarrow A = \frac{4}{\pi}\cdot\left(\frac{1}{4}\cdot\frac{\pi}{4}\right) = \frac{1}{4}.$$

$$f(x) = \sqrt{1-x^2} + \frac{1}{4}x^2.$$

例13. $f(7) = \int_0^7 f'(x)dx$. $(f(0)=0)$.



$$f(7) = \frac{1}{2}\times 2\times 1 = 1$$

例14. $\int_1^3 f(x)dx = \int_1^2 f(x)dx + \int_2^3 f(x)dx$

$$= \int_1^2 f(x)dx + \int_0^1 f(x+2)dx$$

$$= \int_1^2 f(x)dx + \int_0^1 x + f(x)dx$$

$$= \int_0^1 x\,dx + \int_0^2 f(x)dx = \frac{1}{2}.$$

例15. 设 $f(x)$ 原函数为 $\Phi(x)$, $\Phi'(x) = f(x)$

则 $F(t) = \int_1^t \Phi(t) - \Phi(y)\,dy = \Phi(t)\cdot\int_1^t dy - \int_1^t \Phi(y)dy$.

$$F'(t) = \Phi'(t)\cdot(t-1) + \Phi(t) - \Phi(t) = f(t)(t-1)$$

$$F'(2) = f(2)$$

例16. 设 $\int_0^{\pi} f(x)\cos x\,dx = A$.

$$\int_0^{\pi} f\cos x\,dx = \int_0^{\pi} x\cos x\,dx - \int_0^{\pi} A\cos x\,dx$$

$$A = \int_0^{\pi} x\cos x\,dx + 0 = -2$$

$f(x) = x+2$.

$$\int_0^{\pi}(x+2)\sin^4 x\,dx = \int_0^{\pi} x\sin^4 x\,dx + 2\int_0^{\pi}\sin^4 x\,dx$$

$$= \frac{\pi}{2}\int_0^{\pi}\sin^4 x\,dx + 2\int_0^{\pi}\sin^4 x\,dx$$

$$= \left(\frac{\pi}{2}+2\right)\cdot 2\int_0^{\frac{\pi}{2}}\sin^4 x\,dx$$

$$= (\pi+4)\,\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi(\pi+4)}{16}$$

例17. 讨论. $\Delta = 4a^2 - 4(a+2) = 4(a^2-a-2)$

1° $\Delta>0$. $a<-1$ or $a>2$.

两实根. $C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$ 收敛 $\Rightarrow \lambda_1<0, \lambda_2<0$.

$$\begin{cases}\lambda_1+\lambda_2<0\\ \lambda_1\lambda_2>0\end{cases} \Rightarrow \begin{cases}2a<0\\ a+2>0\end{cases} \Rightarrow \begin{cases}a<0\\ a>-2\end{cases} \Rightarrow -2<a<-1$$

2° $\Delta=0$. $a=-1$ or $a=2$.

两相等实根 $(C_1x+C_2)e^{\lambda x} \xRightarrow{\text{收敛}} \lambda<0$.

$$\Rightarrow \begin{cases}\lambda+\lambda<0\\ \lambda^2>0\end{cases} \Rightarrow -2<a<0 \Rightarrow a=-1.$$

3° $\Delta<0$ $-1<a<2$

两虚根 $e^{\alpha x}(C_1\sin\beta x + C_2\cos\beta x)$. 收敛 $\Rightarrow \alpha<0$ ($\alpha=0$ 不收敛).

$$\Rightarrow \begin{cases}2\alpha<0\\ \alpha^2+\beta^2>0\end{cases} \Rightarrow \begin{cases}a<0\\ a+2>0\end{cases} \Rightarrow -2<a<0 \xRightarrow{\Delta<0} -1<a<0$$

全部合起来 $a\in(-2, 0)$.

例18. 齐次通解 $Y = C_1e^{x} + C_2e^{-3x}$

$r^2+2r-3=0,\ r=1,-3$

非齐次特解. $e^{2x}:\ y_1 = ae^{2x}\quad \left(4a+4a-3a=1 \Rightarrow a=\frac{1}{5}\right).$

$3x:\ y_2 = bx+c\quad \left(2b-3bx-3c=3x \Rightarrow b=-1,\ c=\frac{-2}{3}\right).$

$$y = C_1e^{x} + C_2e^{-3x} + \frac{1}{5}e^{2x} - x - \frac{2}{3}.$$

例19. $y_1 = (1+x^2)^2$ 是 $y'+p(x)y = q(x)$ 的解.

$y_2 = \sqrt{1+x^2}$ 是 $y'+p(x)y = 0$ 的解.

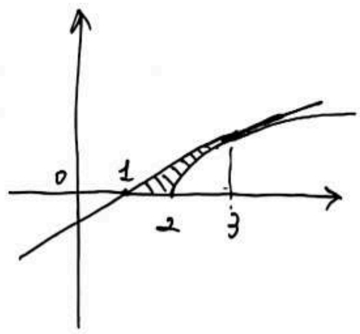
$q(x) = (y_1'+p(x)y_1') - (y_2'+p(x)y_2') \Rightarrow$ 不知道 $p(x)$.

$$p(x) = \frac{y_2'}{y_2} = \frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{1+x^2}} = \frac{x}{1+x^2}.$$

$$\Rightarrow q(x) = \left(4x(1+x^2) - \frac{x}{\sqrt{1+x^2}}\right) + \frac{x}{1+x^2}\left((1+x^2)^2 - \sqrt{1+x^2}\right)$$

$$= 3x(1+x^2).$$

例20. $y' = \frac{1}{2\sqrt{x-2}},\ y'(3) = \frac{1}{2}$, 切线 $y = \frac{1}{2}x - \frac{1}{2}$.



(1) $V_x = \int_1^3 \pi\left(\frac{1}{2}x-\frac{1}{2}\right)^2 dx - \int_2^3 \pi\left(\sqrt{x-2}\right)^2 dx.$

$= \frac{\pi}{4}\int_1^3 x^2-2x+1\,dx - \pi\int_2^3 x-2\,dx.$

$= \frac{\pi}{4}\left(\frac{1}{3}(27-1) - (9-1) + (3-1)\right] - \pi\left(\frac{1}{2}(9-4)-2\right).$

$= \frac{\pi}{6}$

(2) $V_y = \int_1^3 2\pi x\left(\frac{x-1}{2}\right)dx - \int_2^3 2\pi x\sqrt{x-2}\,dx.$

$= \frac{2\pi}{2}\int_1^3 x^2-x\,dx - 2\pi\int_0^1 (t^2+2)\cdot t\cdot 2t\,dt$

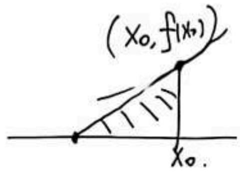
$= \pi\cdot\left(\frac{1}{3}(27-1) - \frac{1}{2}(9-1)\right]$

$-4\pi\int_0^1 t^4+2t^2\,dt$

$= \frac{14}{3}\pi - 4\pi\left(\frac{1}{5}+\frac{2}{3}\right)$

$= \frac{6}{5}\pi.$

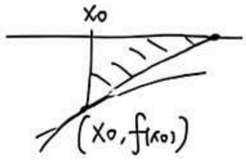
例21.



切线. $y-f(x_0)=f'(x_0)(x-x_0)$.

$y=0$时. $x=x_0-\frac{f(x_0)}{f'(x_0)}$

$$S=\frac{1}{2}|f(x_0)|\cdot\left|\frac{f(x_0)}{f'(x_0)}\right|$$



$$=\frac{1}{2}\frac{f^2(x_0)}{f'(x_0)}\equiv 4.$$

$$f'(x)=\frac{1}{8}f^2(x). \Rightarrow \frac{dy}{dx}=\frac{1}{8}y^2 \Rightarrow \int\frac{8}{y^2}dy=\int dx$$

$$\Rightarrow -\frac{8}{y}=x+C. \Rightarrow (x+c)y+8=0.$$

$$f(0)=2 \Rightarrow 2C+8=0 \Rightarrow C=-4. \Rightarrow f(x)=-\frac{8}{x-4}$$

例22. $S_n=\sum_{j=1}^{n^2}\frac{n}{n^2+j^2}$ $\quad 0\le S_n\le 0$

$n\to\infty \quad S_n=\frac{\pi}{2}$.

类似尝试. $\ln(n+1)\le\sum_{j=1}^{n}\left(\frac{1}{j}\right)\le \ln n+1$.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sin\frac{k}{n}}{n+\frac{1}{k}}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sin\frac{k}{n}}{n+\frac{k}{n}}.$$

$$\int_{-n\pi}^{n\pi}\left[(1+\cos 2x)^{\frac{5}{2}}+\ln\left(x+\sqrt{1+x^2}\right)\right]dx$$

$$\lim_{x\to0^+}\frac{1}{x^4}\int_{\ln(1+x^2)}^{x^2}\frac{e^u-\cos\sqrt{u}}{u}du$$

$f(x)$在$[-2,2]$上有连续的二阶导数, 且$\lim_{x\to0}\frac{f(x)}{x}=1$, 求$\lim_{n\to\infty}\sum_{i=1}^{n}f\left(\frac{i}{n^2}\right)$.