1. $(e^{x+y}-e^{x})dx+(e^{x+y}+e^{y})dy=0$

解: $e^{x}(e^{y}-1)dx+e^{y}(e^{x}+1)dy=0$

$$\frac{e^{x}dx}{e^{x}+1}=-\frac{e^{y}dy}{e^{y}-1}$$

$$\int\frac{de^{x}}{e^{x}+1}=\int-\frac{de^{y}}{e^{y}-1}$$

$$\ln(e^{x}+1)=-\ln|e^{y}-1|+\ln|c|$$

$$(e^{x}+1)(e^{y}-1)=C$$

2. $xy'+y(\ln x-\ln y)=0$

$$\frac{dy}{dx}=\frac{y}{x}(\ln y-\ln x)=\frac{y}{x}\ln\frac{y}{x}$$

令 $\frac{y}{x}=u$ $\quad y=ux \quad \frac{dy}{dx}=\frac{du}{dx}x+u=u\ln u$

$x\frac{du}{dx}=u(\ln u-1)$ 、 $\frac{du}{u(\ln u-1)}=dx$

$$\int\frac{d(\ln u)}{\ln u-1}=\int\frac{1}{x}dx$$

$$\ln|\ln u-1|=\ln|x|+\ln|c|$$

$\ln u-1=cx$ $\quad u=\frac{y}{x}$ 带回 $\quad \ln\frac{y}{x}-1=cx$

3. $(x\frac{dy}{dx}-y)\arctan\frac{y}{x}=x$

$$(\frac{dy}{dx}-\frac{y}{x})\arctan\frac{y}{x}=1$$

令 $\frac{y}{x}=u$ $\quad y=ux \quad \frac{dy}{dx}=\frac{du}{dx}x+u$

$$x\frac{du}{dx}\arctan u=1$$

$$\int\arctan u\,du=\int\frac{dx}{x}$$

$$u\arctan u-\int u\frac{1}{u^{2}+1}du=\int\frac{dx}{x}$$

$$u\arctan u-\frac{1}{2}\ln(u^{2}+1)=\ln|x|+\ln|c|$$

$$u\arctan u=\ln|cx(u^{2}+1)^{\frac{1}{2}}|$$

$cx(u^{2}+1)^{\frac{1}{2}}=e^{u\arctan u}$ $\quad u=\frac{y}{x}$ 带回

$$cx\sqrt{\frac{y^{2}}{x^{2}}+1}=e^{\frac{y}{x}\arctan\frac{y}{x}}$$

$$c\sqrt{x^{2}+y^{2}}=e^{\frac{y}{x}\arctan\frac{y}{x}}$$

4. $y^2 dx - (4xy - 2x^2)\, dy = 0$ "x对齐次"

$$\frac{dx}{dy} = \frac{4xy-2x^2}{y^2} = 4\frac{x}{y} - 2\left(\frac{x}{y}\right)^2 \quad \text{令 } u = \frac{x}{y},\ x = yu \quad \frac{dx}{dy} = u + y\frac{du}{dy}$$

$$u + y\frac{du}{dy} = 4u - 2u^2 \qquad y\frac{du}{dy} = 3u - 2u^2$$

$$\int \frac{du}{3u-2u^2} = \int \frac{1}{y} dy \qquad \left(\frac{1}{3}\cdot\frac{1}{u} + \frac{2}{3}\frac{1}{3-2u} = \frac{1}{u(3-2u)}\right)$$

$$\frac{1}{3}\ln|u| - \frac{1}{3}\ln|3-2u| = \ln|y| + \ln|c|.$$

$\therefore\ \frac{u}{3-2u} = cy^3$、$u = \frac{x}{y}$ 换回

$$\frac{\frac{x}{y}}{3-2\frac{x}{y}} = cy^3 \qquad \therefore\ \frac{x}{3y-2x} = cy^3$$

5. $(2x+y-4)\, dx + (x+y-1)\, dy = 0$

$$\frac{dy}{dx} = -\frac{2x+y-4}{x+y-1} = -\frac{2(x+a)+(y+b)}{(x+a)+(y+b)} \quad \begin{cases} 2a+b=-4 \\ a+b=-1 \end{cases} \quad \begin{cases} a=-3 \\ b=2 \end{cases}$$

令 $Y = y+2,\ X = x-3$

$$\frac{dY}{dX} = \frac{dy}{dx} = -\frac{2X+Y}{X+Y} = -\frac{\frac{Y}{X}+2}{\frac{Y}{X}+1} \quad \text{令 } \frac{Y}{X} = u \quad Y = Xu \quad \frac{dY}{dX} = u + X\frac{du}{dX}$$

$$u + X\frac{du}{dX} = -\frac{u+2}{u+1}$$

$$\frac{u^2+2u+2}{u+1}dX + X\,du = 0$$

$$\frac{u+1}{u^2+2u+2}du = -\frac{dX}{X}$$

$$\int \frac{1}{2}\frac{d(u^2+2u+2)}{u^2+2u+2} = \int -\frac{dX}{X}$$

$$\frac{1}{2}\ln|u^2+2u+2| = -\ln|X| + \ln C$$

$$\sqrt{u^2+2u+2} = \frac{C}{|X|}$$

$C = X^2(u^2+2u+2)$. $u = \frac{Y}{X}$ 带回

$C = X^2\left(\frac{Y^2}{X^2} + 2\frac{Y}{X} + 2\right) = Y^2 + 2XY + 2X^2$. $Y = y+2,\ X = x-3$ 带回

$$2x^2 + 2xy + y^2 - 8x - 2y = C$$

6. $(2x+y-4)\,dx + (2x+y-1)\,dy = 0$

$\frac{dy}{dx} = -\frac{2x+y-4}{2x+y-1}$　　令 $2x+y=u$　　$2+\frac{dy}{dx} = \frac{du}{dx}$

$\frac{du}{dx} - 2 = -\frac{u-4}{u-1}$　　$\frac{du}{dx} = \frac{u+2}{u-1}$

$\int \frac{u+2-3}{u+2}du = \int dx$

$u - 3\ln|u+2| = x + C$　带回

$(2x+y) - 3\ln|2x+y+2| = x + C$

$3\ln|2x+y+2| = x+y+C$

$(2x+y+2)^3 = Ce^{x+y}$

7. $\cos x \frac{dy}{dx} + \sin x \cdot y = 1$、$y(0)=1$

$\frac{dy}{dx} + \tan x\, y = \frac{1}{\cos x}$　　$p(x) = \tan x$、$Q(x) = \frac{1}{\cos x}$

$y = e^{-\int p(x)dx}\left(\int Q(x) e^{\int p(x)dx} dx + C\right)$

$= e^{-\int \tan x\, dx}\left(\int \frac{1}{\cos x} e^{\int \tan x\, dx} dx + C\right)$.　　$\int \tan x\, dx = -\ln|\cos x| + C$

$= e^{\ln|\cos x|}\left(\int \frac{1}{\cos^2 x} dx + C\right)$

$= \cos x\,(\tan x + C)$

令 $x=0$，$y = 1 = C$　$\therefore C = 1$

特解 $y = \sin x + \cos x$

8. $y' = \frac{1}{xy+y^3}$

$\frac{dx}{dy} = xy + y^3$　　$\frac{dx}{dy} - yx = y^3$

$P(y) = -y$，$Q(y) = y^3$

$x = e^{-\int P(y)dy}\left(\int Q(y) e^{\int P(y)dy} dy + C\right)$

$x = e^{\int y\,dy}\left(\int y^3 e^{\int -y\,dy} dy + C\right)$

$x = e^{\frac{y^2}{2}}\left(\int y^3 e^{-\frac{y^2}{2}} dy + C\right)$.

$x = Ce^{\frac{1}{2}y^2} - y^2 - 2$

9T. $y' = \dfrac{1}{xy + x^2y^3}$

$\dfrac{dx}{dy} = xy + x^2y^3 \qquad \dfrac{dx}{dy} - yx = x^2y^3$

$\dfrac{1}{x^2}\dfrac{dx}{dy} - \dfrac{y}{x} = y^3 \qquad d\left(-\dfrac{1}{x}\right) = \dfrac{1}{x^2}dx$

令 $u = -\dfrac{1}{x}$、 $\dfrac{du}{dy} + yu = y^3$, $p(y) = y$、 $Q(y) = y^3$

$u = e^{-\int p(y)dy}\left(\int Q(y) e^{\int p(y)dy} dy + C\right)$

$u = e^{-\frac{y^2}{2}}\left(\int y^3 e^{\frac{y^2}{2}} dy + C\right)$

$u = e^{-\frac{y^2}{2}}\left(y^2 e^{\frac{y^2}{2}} - 2e^{\frac{y^2}{2}} + C\right)$

$u = y^2 - 2 + Ce^{-\frac{y^2}{2}}$

$-\dfrac{1}{x} = y^2 - 2 + Ce^{-\frac{y^2}{2}}$

$\dfrac{1}{x} = Ce^{-\frac{y^2}{2}} - y^2 + 2.$

10. $yy'' + (y')^2 = 0 \qquad y(0) = 1$、$y'(0) = \dfrac{1}{2}$

$y' = p$、 $y'' = \dfrac{dp}{dx} = \dfrac{dp}{dy}\dfrac{dy}{dx} = \dfrac{dp}{dy}y' = \dfrac{dp}{dy}p.$

$y\dfrac{dp}{dy}p + p^2 = 0$

$y\dfrac{dp}{dy} + p = 0 \qquad \int\dfrac{dp}{p} = -\int\dfrac{dy}{y}$

$\ln|p| = -\ln|y| + \ln|C_1|$

$\therefore py = C_1 \quad \therefore y'y = C_1 \quad$ 代值, $C_1 = \dfrac{1}{2}$

$2yy' = 1 \qquad 2y\dfrac{dy}{dx} = 1 \qquad \int 2y\,dx = \int dx \qquad y^2 = x + C_2 \quad$ 代入, $C_2 = 1$

$y^2 = x + 1$. $y = \sqrt{x+1}$

11. $y'' - 3y' + 2y = 2xe^x$

$r^2 - 3r + 2 = 0 \qquad r_1 = 1$、$r_2 = 2 \qquad Y = C_1e^x + C_2e^{2x}$

设 $y^* = x(ax+b)e^x \qquad (y^*)' = (ax^2 + bx + 2ax + b)e^x \qquad (y^*)'' = (ax^2 + (4a+b)x + 2a + 2b)e^x$

令 $(y^*)'' - 3(y^*)' + 2y = 2xe^x$

$ax^2 + (4a+b)x + (2a+2b) - 3[ax^2 + (2a+b)x + b] + 2(ax^2 + bx) = 2x.$

$\begin{cases} a - 3a + 2a = 0 \text{、} 0 = 0 \\ (4a+b) - 3(2a+b) + 2b = 2 \\ (2a+2b) - 3b = 0 \end{cases} \qquad \begin{cases} a = -1 \\ b = -2 \end{cases} \qquad y^* = (-x^2 - 2x)e^x$

$\therefore$ 通解 $y = Y + y^* = C_1e^x + C_2e^{2x} + (-x^2 - 2x)e^x$

12T. $y''+y=x\cos 2x$ $y(0)=1$ $y'(0)=0$

$r^2+1=0$

$r=\pm i$ $(0+i;0-i)$ $Y=e^{0x}(C_1\sin x+C_2\cos x)=C_1\sin x+C_2\cos x$

设 $y^*=(ax+b)\sin 2x+(cx+d)\cos 2x$

求 $(y^*)'$、$(y^*)''$ 代入

$y^*=-\frac{1}{3}x\cos 2x+\frac{4}{9}\sin 2x$

$y=C_1\sin x+C_2\cos x-\frac{1}{3}x\cos 2x+\frac{4}{9}\sin 2x$

$y(0)=1$, $C_2=1$ $y'(0)=0$、$C_1=-\frac{5}{9}$

$y=-\frac{5}{9}\sin x+\cos x-\frac{1}{3}x\cos 2x+\frac{4}{9}\sin 2x$

13T. $y'''-2y''+y'-2y=0$

$r^3-2r^2+r-2=0$

$r^2(r-2)+(r-2)=0$ $(r^2+1)(r-2)=0$

$r_1=i$、$r_2=-i$, $r_3=2$

$y=C_1e^{2x}+C_2\sin x+C_3\cos x$