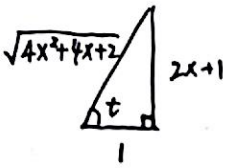
例1： $\int \frac{1}{(2x+1)^3}dx = \frac{1}{2}\int \frac{1}{(2x+1)^3}d(2x+1) = -\frac{1}{4}\frac{1}{(2x+1)^2}+C$. □

例2： $\int \frac{x+1}{(2x^2+2x+1)^2}dx = \int \frac{\frac{1}{4}(4x+2)+\frac{1}{2}}{(2x^2+2x+1)^2}dx = \frac{1}{4}\int \frac{d(2x^2+2x+1)}{(2x^2+2x+1)^2} + \frac{1}{2}\int \frac{1}{(2x^2+2x+1)^2}dx$

$$= -\frac{1}{4}\cdot\frac{1}{(2x^2+2x+1)} + 2\underbrace{\int \frac{1}{[(2x+1)^2+1]^2}dx}_{I_1}$$

在I_1中，令 $2x+1=\tan t$，故 $dx=\frac{1}{2}\sec^2 t\,dt$，$\sin t=\frac{2x+1}{\sqrt{4x^2+4x+2}}$，$\cos t=\frac{1}{\sqrt{4x^2+4x+2}}$



$$\therefore I_1 = 2\int \frac{\frac{1}{2}\sec^2 t}{\sec^4 t}dt = \int \cos^2 t\,dt = \frac{1}{2}\sin t\cdot\cos t+\frac{1}{2}t$$

$$= \frac{1}{2}\cdot\frac{2x+1}{4x^2+4x+2}+\frac{1}{2}\arctan(2x+1)+C$$

故而： $\int \frac{x+1}{(2x^2+2x+1)^2}dx = \frac{x}{4x^2+4x+2}+\frac{1}{2}\arctan(2x+1)+C$. □

例3： $D_n=\int \frac{1}{(x^2+a^2)^n}dx$ 的递推公式.

方法一：如例2，令 $x=a\tan t$. $\therefore D_n=\frac{1}{a^{2n-1}}\int \cos^{2n-2}t\,dt$

$D_{n-1}=\frac{1}{a^{2n-3}}\int \cos^{2n-4}t\,dt$ 两者的关系.

$\sin t=\frac{x}{\sqrt{x^2+a^2}}$

$\cos t=\frac{a}{\sqrt{x^2+a^2}}$

$$D_n=\frac{1}{a^{2n-1}}\int \cos^{2n-3}t\,d\sin t = \frac{1}{a^{2n-1}}\sin t\cdot\cos^{2n-3}t-\frac{1}{a^{2n-1}}\int \sin t\,d\cos^{2n-3}t$$

$$=\frac{1}{a^{2n-1}}\sin t\cdot\cos^{2n-3}t+\frac{2n-3}{a^{2n-1}}\int \cos^{2n-4}t\cdot(1-\cos^2 t)dt$$

$$=\frac{1}{a^2}\cdot\frac{x}{(x^2+a^2)^{n-1}}+(2n-3)a^2\cdot\frac{1}{a^{2n-3}}\int \cos^{2n-4}t\,dt-(2n-3)\cdot\frac{1}{a^{2n-1}}\int \cos^{2n-2}t\,dt$$

$$=\frac{1}{a^2(x^2+a^2)^{n-1}}+(2n-3)a^2D_{n-1}-(2n-3)D_n$$

$$\therefore D_n=\frac{(2n-3)}{(2n-2)a^2}D_{n-1}+\frac{x}{(2n-2)a^2(x^2+a^2)^{n-1}}$$

方法二，对D_{n-1}用分部积分公式：

$$D_{n-1}=\int \frac{1}{(x^2+a^2)^{n-1}}dx = \frac{x}{(x^2+a^2)^{n-1}}-\int x\,d\frac{1}{(x^2+a^2)^{n-1}}$$

$$=\frac{x}{(x^2+a^2)^{n-1}}+(n-1)\int \frac{2x^2+2a^2-2a^2}{(x^2+a^2)^n}dx$$

$$=\frac{x}{(x^2+a^2)^{n-1}}+2(n-1)D_{n-1}-2(n-1)a^2D_n$$

$$\therefore D_n=\frac{(2n-3)}{(2n-2)a^2}D_{n-1}+\frac{x}{(2n-2)a^2(x^2+a^2)^{n-1}}$$ □

例4：留数法.

两边乘 $(x+1)(x+2)(x+3)$：　$x = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$

令 $x=-1$：　$-1 = A\cdot 1\cdot 2$　$\therefore \underline{A=-\frac{1}{2}}$

令 $x=-2$：　$-2 = B\cdot(-1)$　$\therefore \underline{B=2}$

令 $x=-3$：　$-3 = C\cdot(-2)\cdot(-1)$　$\therefore \underline{C=-\frac{3}{2}}$　□

例5：留数法.（有重数情况）

两边乘 $(x+1)^3$：　$2x+1 = A\cdot(x+1)^2 + B(x+1) + C$

令 $x=-1$：　$-1=C$　$\therefore \underline{C=-1}$

求导后再令 $x=-1$：　$2 = 2A(x+1)+B$　$\to$　$\underline{B=2}$

求导后：　$0 = 2A$　$\to$　$\underline{A=0}$.　□

例6：留数法.（有二次因式情况）

两边乘 $x(1+x)(1+x+x^2)$：　$1 = A(x+1)(x^2+x+1) + Bx(x^2+x+1) + (Cx+D)x(x+1)$

分别令 $x=0$，$x=-1$ 可求出：　$\underline{A=1 \quad B=-1}$.

此时：① 由 $\dfrac{Cx+D}{x^2+x+1} = \dfrac{1}{x(x+1)(x^2+x+1)} - \dfrac{1}{x} + \dfrac{1}{x+1}$ 确定出C和D.

② $x^2+x+1=0$ 的根为 $\frac{1}{2}(-1\pm\sqrt{3}i)$

代入 $\frac{1}{2}(-1\pm\sqrt{3}i)$ 得关于C、D的方程组确定出C、D.

总之：求出 $\underline{C=0,\ D=-1}$.　□

例7：$\displaystyle\int\frac{x}{(x-1)(x+1)(x^2+1)}dx = \frac{1}{2}\int\frac{1}{x^4-1}dx^2 = \underline{\frac{1}{4}\ln\left|\frac{x^2-1}{x^2+1}\right| + C}$　□

例8：$\displaystyle\int\frac{x^3}{(x-1)^{100}}dx = \int\frac{(x-1+1)^3}{(x-1)^{100}}dx = \int\frac{1}{(x-1)^{97}}dx + 3\int\frac{1}{(x-1)^{98}}dx + 3\int\frac{1}{(x-1)^{99}}dx + \int\frac{1}{(x-1)^{100}}dx$

$$= \underline{-\frac{1}{96}(x-1)^{-96} - \frac{3}{97}(x-1)^{-97} - \frac{3}{98}(x-1)^{-98} - \frac{1}{99}(x-1)^{-99} + C}.$$ □

例9：$\displaystyle\int\frac{1}{x^4-1}dx = \frac{1}{2}\int\frac{1}{x^2-1}dx - \frac{1}{2}\int\frac{1}{x^2+1}dx = \frac{1}{4}\int\frac{1}{x-1}dx - \frac{1}{4}\int\frac{1}{x+1}dx - \frac{1}{2}\int\frac{1}{x^2+1}dx$

$$= \underline{\frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\arctan x + C}.$$ □

例10：

$$\int \frac{1}{x^4+1}\,dx = \frac{1}{2}\int \frac{x^2+1}{x^4+1}\,dx - \frac{1}{2}\int \frac{x^2-1}{x^4+1}\,dx$$

$$= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx = \frac{1}{2}\int \frac{d(x-\frac{1}{x})}{(x-\frac{1}{x})^2+(\sqrt{2})^2} - \frac{1}{2}\int \frac{d(x+\frac{1}{x})}{(x+\frac{1}{x})^2-(\sqrt{2})^2}$$

$$= \underline{\frac{1}{2\sqrt{2}}\arctan\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) - \frac{1}{4\sqrt{2}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + C} \qquad \square$$

例11：

$$\int \frac{x^2+x}{x^6+1}\,dx = \frac{1}{3}\int \frac{dx^3}{(x^3)^2+1} + \frac{1}{2}\int \frac{dx^2}{(x^2)^3+1}$$

令 $t=x^2$

$$= \frac{1}{3}\arctan(x^3) + \frac{1}{2}\cdot\left(\int \frac{1}{3(t+1)}\,dt - \int \frac{t-2}{3(t^2-t+1)}\,dt\right)$$

$$= \frac{1}{3}\arctan x^3 + \frac{1}{6}\ln|t+1| - \frac{1}{12}\ln|t^2-t+1| + \frac{1}{4}\int \frac{dt}{t^2-t+1}$$

$$= \underline{\frac{1}{3}\arctan x^3 + \frac{1}{12}\ln\left|\frac{(x^2+1)^2}{x^4-x^2+1}\right| + \frac{1}{2\sqrt{3}}\arctan\left(\frac{2x^2-1}{\sqrt{3}}\right) + C}$$

绝对值可以去掉，因为里边始终大于0. $\square$

例12：方法一：(辅助角公式)

$$\int \frac{1}{\sin 2x+1}\,dx = \int \frac{dx}{\sin^2 x+\cos^2 x+2\sin x\cos x} = \int \frac{dx}{(\sin x+\cos x)^2} = \frac{1}{2}\int \csc^2(x+\frac{\pi}{4})\,dx$$

$$= \underline{-\frac{1}{2}\cot(x+\frac{\pi}{4})} + C$$

方法二：(万能公式)

$$\int \frac{1}{\sin 2x+1}\,dx = \int \frac{1+\tan^2 x}{2\tan x+1+\tan^2 x}\,dx = \int \frac{\sec^2 x}{(1+\tan x)^2}\,dx = \int \frac{d\tan x}{(1+\tan x)^2}$$

$$= \underline{-\frac{1}{1+\tan x}} + C \qquad \square$$

例13：

$$\int \frac{1}{\sin^2 x + a\cos^2 x}\,dx = \int \frac{1/\cos^2 x}{\tan^2 x + a}\,dx = \int \frac{d\tan x}{\tan^2 x + a}.$$

① 若 $a=0$，则 $\underline{I = -\frac{1}{\tan x}} + C$.

② 若 $a>0$，则 $\underline{I = \frac{1}{\sqrt{a}}\arctan\frac{x}{\sqrt{a}}}$.

③ 若 $a<0$，则 $\underline{I = \frac{1}{2\sqrt{a}}\ln\left|\frac{x-\sqrt{a}}{x+\sqrt{-a}}\right|}$. $\square$

例14： $\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x = 1 - \frac{1}{2}\sin^2 2x$.

$$\therefore I = \int \frac{\sin x\cos x}{1-\frac{1}{2}\sin^2 2x}dx = \frac{1}{2}\int \frac{\sin 2x \cdot d2x}{2-\sin^2 2x} = -\frac{1}{2}\int \frac{d\cos 2x}{\cos^2 2x + 1} = \underline{-\frac{1}{2}\arctan(\cos 2x) + C}$$ □

例15： $\sin^4 x + \cos^4 x = 1 - \frac{1}{2}\sin^2 2x$

$$\therefore I = \int \frac{dx}{1-\frac{1}{2}\sin^2 2x} = \cancel{\frac{1}{2}}\int \frac{2}{\sin^2 2x + 2\cos^2 2x}dx = \cancel{\frac{1}{4}\int \frac{d\tan 2x}{\tan^2 2x + 2}}$$

$$= \int \frac{d\tan 2x}{\tan^2 2x + 2} = \underline{\frac{1}{\sqrt{2}}\arctan\left(\frac{\tan 2x}{\sqrt{2}}\right) + C}$$ □

例16： $\sin^6 x + \cos^6 x = (\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x - \sin^2 x\cos^2 x)$

$$= 1 - \frac{3}{4}\sin^2 2x$$

$$\therefore I = \int \frac{dx}{1-\frac{3}{4}\sin^2 2x} = 2\int \frac{d2x}{\sin^2 2x + 4\cos^2 2x} = \underline{\arctan\left(\frac{\tan 2x}{2}\right) + C}$$ □

例17： $n=3$时： $\sqrt{x\sqrt{x\sqrt{x}}} = \sqrt{x\sqrt{x\cdot x^{\frac{1}{2}}}} = \sqrt{x\cdot x^{\frac{3}{4}}} = x^{\frac{7}{8}}$.

$\therefore$ 对一般的n而言： $\underbrace{\sqrt{x\sqrt{\cdots\sqrt{x}}}}_{n} = x^{1-\frac{1}{2^n}}$.

$$\therefore I = \int x^{1-\frac{1}{2^n}}dx = \underline{\frac{1}{2-\frac{1}{2^n}}x^{2-\frac{1}{2^n}} + C}$$ □

例18：

$$\int \frac{m\cos x + n\sin x + l}{\sin x + \cos x}dx = l\int \frac{1}{\sin x + \cos x}dx + \frac{m+n}{2}\int \frac{\sin x + \cos x}{\sin x + \cos x}dx + \frac{m-n}{2}\int \frac{\cos x - \sin x}{\sin x + \cos x} dx$$

$$= \underline{\frac{l}{\sqrt{2}}\ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{8}\right)\right| + \frac{m+n}{2}x + \frac{m-n}{2}\ln|\sin x + \cos x| + C}$$

$$\frac{m-n}{2}\int \frac{\cos x - \sin x}{\sin x + \cos x}dx = \frac{m-n}{2}\int \frac{d(\sin x + \cos x)}{\sin x + \cos x}$$ □

例19&例20：

$$\begin{cases} \int \sin(\ln x)dx = x\sin(\ln x) - \int x\, d\sin(\ln x) = x\sin(\ln x) - \int \cos(\ln x)dx \\ \int \cos(\ln x)dx = x\cos(\ln x) + \int \sin(\ln x)dx \end{cases}$$

解得：

$$\begin{cases} \int \sin(\ln x)dx = \underline{\frac{1}{2}x[\sin(\ln x) - \cos(\ln x)] + C} \\ \int \cos(\ln x)dx = \underline{\frac{x}{2}[\sin(\ln x) + \cos(\ln x)] + C} \end{cases}$$ □

例21：　$\because$ $e^{-|x|}$ 在 $\mathbb{R}$ 上连续，则 $\int e^{-|x|}dx$ 存在且连续．

$x\geq 0$ 时：$\int e^{-|x|}dx=\int e^{-x}dx=-e^{-x}+C_1$

$x<0$ 时：$\int e^{-|x|}dx=\int e^{x}dx=e^{x}+C_2$

但由于 $\int e^{-|x|}dx$ 连续，因此 C_1，C_2 之间有关系：$C_2=C_1-2$．

即令：$F(x)=\begin{cases}-e^{-x}+1 & x\geq 0\\ e^{x}-1 & x<0\end{cases}$

则 $\int e^{-|x|}dx=F(x)+C$．（C 任意）． $\square$