1T. $\int x\arctan x\,dx$

$= \frac{1}{2}\int \arctan x\ d(x^2+1)$

$= \frac{1}{2}(x^2+1)\arctan x - \frac{1}{2}\int (x^2+1)\ d(\arctan x)$

$= \frac{1}{2}(x^2+1)\arctan x - \frac{1}{2}\int (x^2+1)\frac{1}{x^2+1}\,dx$

$= \frac{1}{2}(x^2+1)\arctan x - \frac{1}{2}x + C$

1T.* $\int x\ln(1+x^2)\arctan x\,dx$

$= \frac{1}{2}\int \ln(1+x^2)\arctan x\,d(x^2+1)$

$= \frac{1}{2}(x^2+1)\ln(1+x^2)\arctan x - \frac{1}{2}\int (x^2+1)\left[\frac{2x}{x^2+1}\arctan x + \ln(1+x^2)\frac{1}{x^2+1}\right]dx$

$= \frac{1}{2}(x^2+1)\ln(1+x^2)\arctan x - \int x\arctan x\,dx - \frac{1}{2}\int \ln(1+x^2)\,dx$

对于 $\int x\arctan x\,dx$，见 1

对于 $\int \ln(1+x^2)\,dx = x\ln(1+x^2) - \int \frac{2x^2}{x^2+1}\,dx = x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx$

$= x\ln(1+x^2) - 2\int \frac{x^2+1-1}{1+x^2}\,dx = x\ln(1+x^2) - 2x + 2\arctan x$

$\therefore$ 原式 $= \frac{1}{2}\arctan x\left[(x^2+1)\ln(x^2+1) - (1+x^2)\right] - \frac{1}{2}x\ln(1+x^2) + \frac{3}{2}x - \arctan x + C$

2T. $\int x^3 e^x\,dx$

补充：$\int x^3\cos ax\,dx$ 也可以用这种方法

求导	x^3	$3x^2$	$6x$	6	0 $\int 0\,dx$
积分	e^x	e^x	e^x	e^x	e^x

$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x$

正常分部：原 $= \int x^3\,de^x = x^3e^x - 3\int e^x x^2\,dx = x^3e^x - 3\int x^2\,de^x$

$= x^3e^x - 3\left[x^2e^x - 2\int e^x x\,dx\right] = x^3e^x - 3x^2e^x + 6\int x\,de^x$

$= x^3e^x - 3x^2e^x + 6\left(xe^x - \int e^x\,dx\right) = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$

3T. $\int e^x\sin x\,dx$ 令 $I=$ 原式

求导	e^x	e^x	e^x	$\int -e^x\sin x\,dx$
积分	$\sin x$	$-\cos x$	$-\sin x$	

或

求导	$\sin x$	$\cos x$	$-\sin x$
积分	e^x	e^x	e^x

$I = -e^x\cos x + e^x\sin x - I$

$I = e^x\sin x - e^x\cos x - I$

$\Rightarrow I = \frac{1}{2}e^x(\sin x - \cos x) + C$

若不用表格法：

$\int e^x \sin x\,dx = \int \sin x\,de^x = e^x \sin x - \int e^x \cos x\,dx$

$= e^x \sin x - \int \cos x\,de^x = e^x \sin x - e^x \cos x + \int e^x(-\sin x)\,dx$ ✓

4T. $\int \sqrt{\frac{x}{x+1}}\,dx$ 令 $t=\sqrt{\frac{x}{x+1}}$ $\Rightarrow x=\frac{1}{1-t^2}-1$ $dx=d\left(\frac{1}{1-t^2}\right)\neq\frac{-2t\,dt}{(1-t^2)^2}$

$=\int t\,d\left(\frac{1}{1-t^2}\right)$

$=\frac{t}{1-t^2}-\int\frac{1}{1-t^2}dt=\frac{t}{1-t^2}+\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+C=\frac{\sqrt{\frac{x}{x+1}}}{1-\frac{x}{x+1}}+\frac{1}{2}\ln\left|\frac{\sqrt{\frac{x}{x+1}}-1}{\sqrt{\frac{x}{x+1}}+1}\right|+C$

5T. $\int\frac{1}{x}\sqrt{\frac{x+1}{x}}\,dx$ 令 $\sqrt{\frac{x+1}{x}}=t>1$ $\Rightarrow x=\frac{1}{t^2-1}$ $dx=d\left(\frac{1}{t^2-1}\right)=\frac{-2t\,dt}{(t^2-1)^2}$

$=\int(t^2-1)t\,d\left(\frac{1}{t^2-1}\right)$

法一：拆开 $=-\int\frac{2t^2}{t^2-1}dt=-2\int\frac{t^2-1+1}{t^2-1}dt=-\left(2t+\ln\left|\frac{t-1}{t+1}\right|\right)+C$

法二：不拆开 $=t-\int\frac{1}{t^2-1}d(t^3-t)=t-\int\frac{3t^2-1}{t^2-1}dt=t-\int\frac{3(t^2-1)+2}{t^2-1}dt$

$=t-3t-\ln\left|\frac{t-1}{t+1}\right|+C=-2t-\ln\left|\frac{t-1}{t+1}\right|+C$ 再带回

6T. $\int\left(\sqrt{\frac{1-x}{1+x}}+\sqrt{\frac{1+x}{1-x}}\right)dx$

法一：$I=\int\left(\frac{1-x}{\sqrt{1-x^2}}+\frac{1+x}{\sqrt{1-x^2}}\right)dx=2\int\frac{1}{\sqrt{1-x^2}}dx=2\arcsin x+C$

法二：分开算，这里只算 $\int\sqrt{\frac{1-x}{1+x}}dx$

$=\int\frac{1-x}{\sqrt{1-x^2}}dx=\int\frac{1}{\sqrt{1-x^2}}dx-\frac{x}{\sqrt{1-x^2}}dx=\arcsin x+\frac{1}{2}\frac{d(1-x^2)}{\sqrt{1-x^2}}=\arcsin x+\sqrt{1-x^2}+C$

法三：$\sqrt{\frac{1-x}{1+x}}=t$ $x=\frac{t^2+1}{1-t^2}$ 比较复杂

7T. $\int\frac{xe^x}{\sqrt{e^x-2}}dx$

令 $t=\sqrt{e^x-2}$ $t^2=e^x-2$ $x=\ln(t^2+2)$ $dx=d(\ln(t^2+2))=\frac{2t}{t^2+2}dt$

原式 $=\int\frac{\ln(t^2+2)\cdot(t^2+2)}{t}\frac{2t}{t^2+2}dt$

$=2\int\ln(t^2+2)\,dt$

$=2t\ln(t^2+2)-2\int t\frac{2t}{t^2+2}dt$

$=2t\ln(t^2+2)-4\int\frac{t^2+2-2}{t^2+2}dt$

$=2t\ln(t^2+2)-4t+8\frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}}+C$

$=2\sqrt{e^x-2}\,x-4\sqrt{e^x-2}+8\frac{1}{\sqrt{2}}\arctan\frac{\sqrt{e^x-2}}{\sqrt{2}}+C$

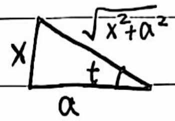
8T. $I=\int \frac{x^2}{(a^2+x^2)^2}dx$

法一: $\cancel{令}$ $x=a\tan t$ $dx=a\sec^2 t\,dt$ $t=\arctan\frac{x}{a}$

$$I=\int \frac{a^2\tan^2 t}{a^4\sec^4 t}a\sec^2 t\,dt=\frac{1}{a}\int\frac{\tan^2 t}{\sec^2 t}dt=\frac{1}{a}\int \sin^2 t\,dt=\frac{1}{a}\int\frac{1-\cos 2t}{2}dt.$$

$$=\frac{1}{2a}\left[t-\frac{1}{2}\sin 2t\right]+C$$

☆ $$=\frac{1}{2a}\left[\arctan\frac{x}{a}-\sin t\cos t\right]+C$$



$\sin t=\frac{x}{\sqrt{x^2+a^2}}$

$\cos t=\frac{a}{\sqrt{x^2+a^2}}$

禁止写 $\sin\arctan\frac{x}{a}$, $\cos\arctan\frac{x}{a}$!

$$=\frac{1}{2a}\arctan\frac{x}{a}-\frac{1}{2}\frac{x}{x^2+a^2}+C$$

法二: $$I=\int\frac{x\cdot x\,dx}{(a^2+x^2)^2}=\frac{1}{2}\int\frac{x\cdot d(x^2)}{(a^2+x^2)^2}=\frac{1}{2}\int\frac{x\,d(a^2+x^2)}{(a^2+x^2)^2}=-\frac{1}{2}\int x\,d\left(\frac{1}{a^2+x^2}\right)$$

思考角度二:

$\left(-\frac{1}{u}\right)'=\frac{1}{u^2}$, 那么令 $d\left(\frac{1}{a^2+x^2}\right)=\frac{-2x}{(a^2+x^2)^2}$, 再凑成

原式 $$=-\frac{x}{2}\frac{1}{a^2+x^2}+\frac{1}{2}\int\frac{1}{a^2+x^2}dx=-\frac{x}{2}\frac{1}{a^2+x^2}+\frac{1}{2a}\arctan\frac{x}{a}+C$$

9T. $\int e^{2x}\arctan\sqrt{e^x-1}\,dx$

令 $t=\sqrt{e^x-1}$ $t^2=e^x-1$ $x=\ln(t^2+1)$ $dx=\frac{2t}{t^2+1}dt$

$$I=\int(t^2+1)^2\arctan t\,\frac{2t}{t^2+1}dt$$
$$=\int(t^2+1)\arctan t\,2t\,dt$$
$$=\int(t^2+1)\arctan t\,d(t^2+1)$$
$$=\frac{1}{2}\int\arctan t\,d(t^2+1)^2$$
$$=\frac{1}{2}(t^2+1)^2\arctan t-\frac{1}{2}\int(t^2+1)^2\frac{1}{t^2+1}dt$$
$$=\frac{1}{2}(t^2+1)^2\arctan t-\frac{t^3}{6}-\frac{t}{2}+C$$

10T. $\int\ln\left(1+\sqrt{\frac{1+x}{x}}\right)dx$

令 $t=\sqrt{\frac{1+x}{x}}$ $dx=d\left(\frac{1}{t^2-1}\right)$

$$I=\int\ln(1+t)\,d\left(\frac{1}{t^2-1}\right)$$
$$=\frac{\ln(1+t)}{t^2-1}-\int\frac{1}{(t^2-1)(t+1)}dt$$ ✓

11T. $\int \frac{xe^x \, dx}{(1+x)^2}$

$= -\int xe^x \, d\left(\frac{1}{1+x}\right)$

$= -\frac{xe^x}{1+x} + \int \frac{1}{1+x} (1+x)e^x \, dx$

$= -\frac{xe^x}{1+x} + e^x + C$

11T * $\int \frac{xe^x}{(1+e^x)^2} dx$ $\qquad \left(\frac{1}{1+e^x}\right)' = -\frac{e^x}{(1+e^x)^2}$

$= -\int x \, d\left(\frac{1}{1+e^x}\right)$

$= \frac{-x}{1+e^x} + \underline{\int \frac{1}{1+e^x} dx}$ ☆

对 $\int \frac{1}{1+e^x} dx$ 法一: $\int \frac{1}{1+e^x} dx = \int \frac{e^x \, dx}{(1+e^x)e^x} = \int \frac{de^x}{e^x(e^x+1)} = \int \frac{de^x}{e^x} - \int \frac{d(e^x+1)}{e^x+1}$

$= \ln e^x - \ln(e^x+1) = \ln \frac{e^x}{e^x+1}$

法二: $\int \frac{1}{1+e^x} dx = \int \frac{1+e^x-e^x}{1+e^x} dx = x - \int \frac{d(e^x+1)}{1+e^x} = x - \ln(e^x+1)$.

$\therefore$ 原式 $= -\frac{x}{1+e^x} + x - \ln(e^x+1) + C$.

12T. $\int \frac{dx}{x(x^{10}+1)}$

法一: 令 $\frac{1}{x} = t$ $\quad x = \frac{1}{t}$ $\quad dx = d\left(\frac{1}{t}\right) = -\frac{1}{t^2} dt$

$I = \int \frac{-\frac{1}{t^2} dt}{\frac{1}{t}\left(\frac{1}{t^{10}}+1\right)} = -\int \frac{t^9 \, dt}{1+t^{10}} = -\frac{1}{10}\int \frac{dt^{10}}{1+t^{10}} = -\frac{1}{10}\int \frac{d(1+t^{10})}{1+t^{10}} = -\frac{1}{10}\ln(1+t^{10}) + C$

$= -\frac{1}{10}\ln\left(1+\frac{1}{x^{10}}\right) + C$

法二: $\int f(x^n) \frac{1}{x} dx = \frac{1}{n}\int f(x^n) \frac{1}{x^n} d(x^n)$

$I = \int \frac{x^9 \, dx}{x^{10}(x^{10}+1)} = \frac{1}{10}\int \frac{dx^{10}}{x^{10}(x^{10}+1)} = \frac{1}{10} \int \frac{dx^{10}}{x^{10}} - \int \frac{d(x^{10}+1)}{x^{10}+1} = \frac{1}{10}\left(\ln x^{10} - \ln(x^{10}+1)\right) + C$

13T. $\int \frac{x+1}{x(1+xe^x)} dx$

对复杂因式求导: $(xe^x)' = (x+1)e^x$, 故分子分母同乘 x

$\int \frac{(x+1)e^x}{xe^x(1+xe^x)} dx = \int \frac{d(xe^x)}{xe^x(1+xe^x)} = \int \frac{d(xe^x)}{xe^x} - \int \frac{d(1+xe^x)}{1+xe^x}$

$= \ln(xe^x) - \ln(1+xe^x)$

13T * $\int \frac{1+x\cos x}{x(1+xe^{\sin x})} dx$

$(xe^{\sin x})' = e^{\sin x} + xe^{\sin x}\cos x = e^{\sin x}(1+x\cos x)$

$I = \int \frac{e^{\sin x}(1+x\cos x) \, dx}{e^{\sin x} x(1+xe^{\sin x})} = \int \frac{d(xe^{\sin x})}{(xe^{\sin x}+1)xe^{\sin x}}$