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1. $y=\ln^3(\sin^2x+1)$, $y'=3\ln^2(\sin^2x+1)\cdot\frac{1}{\sin^2x+1}\cdot 2\sin x\cdot\cos x=\frac{3\sin 2x}{\sin^2x+1}\ln^2(\sin^2x+1)$.

2、数学归纳法.

$n=1$时,有 $(e^{\frac{1}{x}})'=-\frac{1}{x^2}e^{\frac{1}{x}}$,满足命题.

假设 $n=m$ 时,有 $(x^{m-1}e^{\frac{1}{x}})^{(m)}=e^{\frac{1}{x}}\cdot\frac{(-1)^m}{x^{m+1}}$ 成立.

则 $n=m+1$ 时. $(x^{m}e^{\frac{1}{x}})^{(m+1)}=(x\cdot x^{m-1}e^{\frac{1}{x}})^{(m+1)}$ ……出现归纳假设中有的结构.

令 $u=x$. $v=x^{m-1}e^{\frac{1}{x}}$ 则.

$$(x^m e^{\frac{1}{x}})^{(m+1)}=\sum_{k=0}^{m+1}C_{m+1}^{k}u^{(k)}v^{(m+1-k)}=uv^{(m+1)}+C_{m+1}^{1}u'v^{(m)}$$

$$=x(x^{m-1}e^{\frac{1}{x}})^{(m+1)}+(m+1)(x^{m-1}e^{\frac{1}{x}})^{(m)}$$

$$=x\left[e^{\frac{1}{x}}\cdot\frac{(-1)^m}{x^{m+1}}\right]'+(m+1)\cdot e^{\frac{1}{x}}\cdot\frac{(-1)^m}{x^{m+1}}$$

$$=x\left[-e^{\frac{1}{x}}\frac{(-1)^m}{x^{m+3}}+\cancel{e^{\frac{1}{x}}\frac{(-1)^{m+1}(m+1)}{x^{m+2}}}\right]+\cancel{(m+1)e^{\frac{1}{x}}\frac{(-1)^m}{x^{m+1}}}$$

$$=\frac{(-1)^{m+1}e^{\frac{1}{x}}}{x^{m+2}}$$,命题成立.

故对任意n为正整数,均有 $(x^{n-1}e^{\frac{1}{x}})^{(n)}=e^{\frac{1}{x}}\frac{(-1)^n}{x^{n+1}}$ 成立.

3、记: $\begin{cases}x=t+\sin t\triangleq\psi(t)\\ y=\arctan(t-1)\triangleq\varphi(t)\end{cases}$,则 $\begin{cases}\frac{dx}{dt}=1+\cos t\\ \frac{dy}{dt}=\frac{1}{1+(t-1)^2}=\frac{1}{t^2-2t+2}\end{cases}$.

则 $\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{dy}{dt}\cdot\left(\frac{dx}{dt}\right)^{-1}=\frac{1}{(t^2-2t+2)(1+\cos t)}$

则 $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\left(\frac{dx}{dt}\right)^{-1}=\frac{d}{dt}\left[\frac{1}{(t^2-2t+2)(1+\cos t)}\right]\cdot\frac{1}{1+\cos t}$ ×××

发现很难计算

$\frac{dy}{dx}=\frac{dy}{dt}\cdot\left(\frac{dx}{dt}\right)^{-1}=\frac{\varphi'(t)}{\psi'(t)}$, $\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{\varphi'(t)}{\psi'(t)}\right)\cdot\frac{1}{\psi'(t)}=\frac{\varphi''(t)\psi'(t)-\psi''(t)\varphi'(t)}{\psi'^3(t)}$

由 $\begin{cases}\psi'(t)=1+\cos t\\ \varphi'(t)=\frac{1}{t^2-2t+2}\end{cases}$ $\begin{cases}\psi''(t)=-\sin t\\ \varphi''(t)=-\frac{2t-2}{(t^2-2t+2)^2}\end{cases}$ $\Rightarrow$ $\begin{cases}\psi'(0)=2\\ \psi''(0)=0\\ \varphi'(0)=\frac{1}{2}\\ \varphi''(0)=\frac{1}{2}\end{cases}$

则 $\left.\frac{d^2y}{dx^2}\right|_{t=0}=\frac{\frac{1}{2}\times2}{2^3}=\frac{1}{8}$.

4. $y = \frac{5}{2+3x-2x^2}$ 　 $\begin{matrix} -2 & 1 \\ 1 & -2 \end{matrix}$

$= \frac{5}{(-2x-1)\cdot(x-2)}$ …… 利用真分式的结论拆分分式。~~裂项~~

令 $y = \frac{A}{-2x-1} + \frac{B}{x-2}$，

$\frac{5}{-2x-1} = \frac{A(x-2)}{-2x-1} + B \xrightarrow{x=2} B = -1$.

$\frac{5}{x-2} = A + \frac{B(-2x-1)}{x-2} \xrightarrow{x=-\frac{1}{2}} A = -2$.

则 $y = \frac{1}{x+\frac{1}{2}} - \frac{1}{x-2}$，可得 $y^{(n)} = \frac{(-1)^n n!}{(x+\frac{1}{2})^{n+1}} - \frac{(-1)^n n!}{(x-2)^{n+1}}$.

5. 「思考：求某一特定点的导数，该怎么办？

注意多项式的一个性质：$(x-a)^n$ 在 $x=a$ 的 k 阶导数仅有 $k=n$ 时为 $n!$，其余均0。

$\rightsquigarrow u = (x+1)^9 \quad v = \cdots$

令 $u = (x-1)^9$，$v = (x^2+x+1)^9 e^{2x}$

则 $(uv)^{(10)}\Big|_{x=1} = \sum_{k=0}^{10} C_{10}^{k} u^{(10-k)} v^{(k)}\Big|_{x=1} = C_{10}^{9}\, u^{(9)}\Big|_{x=1} v'\Big|_{x=1} = 10\cdot u^{(9)}\Big|_{x=1} v'\Big|_{x=1}$.

又 $u^{(9)}\Big|_{x=1} = 9!$，$v' = [(x^2+x+1)^9 e^{2x}]' = [9(x^2+x+1)^8(2x+1) + 2(x^2+x+1)^9]e^{2x}$.

则 $v'\Big|_{x=1} = [9\times 3^9 + 2\times 3^9]e^2 = 11\times 3^9 e^2$.

则 $[(x^3-1)^9 e^{2x}]^{(10)}\Big|_{x=1} = 10\times 9!\times 11\times 3^9 e^2 = 110\times 9!\times 3^9 e^2$.

6. $x^y = y^x \Rightarrow x\ln y = y\ln x \Rightarrow \ln y + x\cdot\frac{y'}{y} = y'\ln x + \frac{y}{x}$.

~~$\Rightarrow (\ln x - \frac{x}{y})\, y' = \ln y - \frac{y}{x} \Rightarrow y' = \frac{xy\ln y - y^2}{xy\ln x - y^2}$~~

$\Rightarrow (\ln x - \frac{x}{y})y' = \ln y - \frac{y}{x} \Rightarrow y' = \frac{xy\ln y - y^2}{xy\ln x - x^2}$.

7. (1) $f'(x) = \frac{1}{1+x^2} \Rightarrow (1+x^2)f'(x) = 0$. 求n阶导 $n\geq 2$

令 $u = 1+x^2$，$v = f'(x)$，求 $n-1$ 阶导，$n\geq 3$，则

$\sum_{k=0}^{n-1} C_{n-1}^{k} u^{(k)} v^{(n-1-k)} = 0$，即 $C_{n-1}^{0} uv^{(n-1)} + C_{n-1}^{1} u'v^{(n-2)} + C_{n-1}^{2} u''v^{(n-3)} = 0$.

也即 $(1+x^2)f^{(n)}(x) + 2(n-1)x\, f^{(n-1)}(x) + \frac{n(n-1)}{2}\cdot 2 f^{(n-2)}(x) = 0$.

令 $x=0$，则 $f^{(n)}(0) = -n(n-1)f^{(n-2)}(0)$，$\forall n\geq 3$，

又 $f^{(1)}(0) = \frac{1}{1+x^2}\Big|_{x=0} = 1$. 　$f^{(2)}(0) = -\frac{2x}{(1+x^2)^2}\Big|_{x=0} = 0$. 　$f^{(2)}(0) = 0$.

① 当 n 为偶数时，$f^{(n)}(0) = -n(n-1)f^{(n-2)}(0) = \cdots = -n(n-1)(n-2)\cdots 3\times 2 f^{(2)}(0) = 0$.

② 当 n 为奇数时，设 $n = 2k+1$，$k\geq 0$

$f^{(2k+1)}(0) = -2k(2k-1)f^{(2k-1)}(0) = (-1)^2 2k(2k-1)(2k-2)(2k-3)f^{(2k-3)}(0)$

$= \cdots = (-1)^k (2k)!\, f'(0) = (-1)^k (2k)!$

(2) $g'(x) = \frac{1}{\sqrt{1-x^2}} \Rightarrow (1-x^2)g'^2(x) = 0 \Rightarrow -2xg'^2(x) + (1-x^2)g'(x)g''(x) = 0$ 即有：

$-2xg'(x) + (1-x^2)g''(x) = 0$，求n阶导、留作习题. 答案：$\begin{cases} (2m-1)!!, & 当 n=2m+1 \\ 0, & 当 n=2m \end{cases}$