① A. $f(x)$ 在0附近2阶可导: $f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}\cdot x^2+o(x^2)$

$\lim\limits_{x\to0}\frac{f(x)}{x^2}=\lim\limits_{x\to0}\frac{f(0)+f'(0)x+\frac{f''(0)}{2}\cdot x^2+o(x^2)}{x^2}=1 \Rightarrow f(0)=0,\ f'(0)=0,\ f''(0)=2$. 正确

注: 若条件改为在0某邻域有一阶导数. 则A选项错误. 反例: $f(x)=\begin{cases}0 & x=0\\ x^2+x^3\sin\frac{1}{x} & x\neq0\end{cases}$

B.C.D 由保号性. $\exists\, x_0=0$ 邻域 $\frac{f(x)}{x^2}>0 \Rightarrow f(x)>0=f(0)$. 取极小值.

答案 A.C.

② $f(x)=f(x_0)+\frac{f'''(x_0)}{3!}(x-x_0)^3+o((x-x_0)^3) \Rightarrow \frac{f(x)-f(x_0)}{(x-x_0)^3}=\frac{f'''(x_0)}{3!}+\frac{o((x-x_0)^3)}{(x-x_0)^3}\to\frac{f'''(x_0)}{3!}$

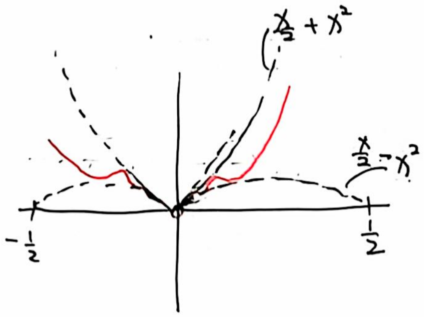
由保号性: $x>x_0$: $\frac{f(x)-f(x_0)}{(x-x_0)^3}>0 \Rightarrow f(x)>f(x_0)$ $x<x_0$: …… $f(x)<f(x_0)$. 不是极值点.

同理: $f''(x)=\underset{0}{\underset{\|}{f''(x_0)}}+f'''(x_0)(x-x_0)+o(x-x_0) \Rightarrow \frac{f''(x)}{x-x_0}=f'''(x_0)+o(x-x_0)\to f'''(x_0)>0$

由保号性: $x>x_0$: $f''(x)>0$, $x<x_0$: $f''(x)<0$, $(x_0,f(x_0))$是拐点.

答案 C.

③ 错. 如: $f(x)=\begin{cases}\frac{x}{2}+x^2\sin\frac{1}{x} & x>0\\ 0 & x=0\\ \frac{-x}{2}+x^2\sin\frac{-1}{x} & x<0\end{cases}$ 对称.



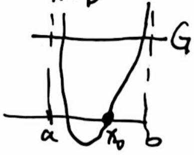
$x=0$ 是极小值点. 但不是单调性分界点.

④ (Rolle Thm 的推广)
*若 $\lim\limits_{x\to a^+}f(x)=\lim\limits_{x\to b^-}f(x)=A$ 则通过补充定义 $\tilde f(x)=\begin{cases}A & x=a\\ f(x) & a<x<b\\ B & x=b\end{cases}$ $\tilde f(x)\in C[a,b]$.

则$\exists\,\xi\in(a,b)$ s.t. $\tilde f'(\xi)=f'(\xi)=0$

对于 $\lim\limits_{x\to a^+}f(x)=\lim\limits_{x\to b^-}f(x)=+\infty$

I 若有零点.



$\forall G>0\ \exists\,\delta\ (a,a+\delta)\ f(x)>G$ 取x_1 $[x_1,x_0]$内用介值Thm. $\exists f(x')=G$.
同理 $[x_0,x_2]$内. $\exists f(x'')=G$. 在$[x',x'']$用Rolle Thm. 即可.

若无零点. $\tilde f(x)=\begin{cases}0 & x=a\\ \frac{1}{f(x)} & \\ 0 & x=b\end{cases}$ 则$\exists\,\xi\in(a,b)$ $\tilde f'(\xi)=-\frac{1}{f^2(\xi)}\cdot f'(\xi)=0$

II $\tilde f(x)=\begin{cases}\frac{\pi}{2} & x=a\\ \arctan f(x) & a<x<b\\ \frac{\pi}{2} & x=b\end{cases}$ 则$\exists\,\xi\in(a,b)$ $\tilde f'(x)=\frac{f'(\xi)}{1+f(\xi)^2}=0$

⑤ I. $f'(0)=\lim_{x\to0}\frac{\frac{g(x)}{x}-0}{x-0}=\lim_{x\to0}\frac{g(x)}{x^2}\Leftarrow\lim_{x\to0}\frac{g'(x)}{2x}=\lim_{x\to0}\frac{g'(x)-g'(0)}{2(x-0)}=\frac{1}{2}g''(0)=5$

$f''(0)$存在,则$f(x)$, $f'(x)$在0邻域内连续,可用L'Hospital法则.

II $\lim_{x\to0}\frac{g(x)}{x^2}=\lim_{x\to0}\frac{g(0)+g'(0)x+g''(0)\frac{x^2}{2}+o(x^2)}{x^2}=5.$

⑥ 求导=0, 代入 $x=1$.

⑦ 在 $x=0,1$ 的Taylor公式 (由系数4联想到代入 $x=\frac{1}{2}$). $f(0)+$

$$\left.\begin{aligned}&f(x)=f(0)+\underset{0}{\underset{\|}{f'(0)}}x+\frac{f''(\xi_1)}{2!}x^2\Rightarrow f(\tfrac{1}{2})=\frac{f''(\xi_1)}{2}\cdot\frac{1}{4}\\&f(x)=f(1)+\underset{0}{\underset{\|}{f'(1)}}(x-1)+\frac{f''(\xi_2)}{2!}(x-1)^2\Rightarrow f(\tfrac{1}{2})=f(1)+\frac{f''(\xi_2)}{2}\cdot\frac{1}{4}\end{aligned}\right\}$$

$$\Rightarrow 4|f(1)-f(0)|=\frac{1}{2}|f''(\xi_1)-f''(\xi_2)|\le\frac{|f''(\xi_1)|+|f''(\xi_2)|}{2}\le\underset{|f''(\xi)|.}{\underset{\|}{\max\{|f''(\xi_1)|,|f''(\xi_2)|\}}}.$$

故取ξ为 ξ_1,ξ_2 中 $|f(\xi_i)|$最大者.

⑧ 由ln x 连续性: $\lim_{x\to0}\frac{\ln(1+x+\frac{f(x)}{x})}{x}=3$ $\quad \overset{*}{\lim_{x\to0}}e^{\left(\frac{\ln(1+x+\frac{f(x)}{x})}{x}\cdot x\right)}=\lim_{x\to0}(1+x+\frac{f(x)}{x})=1$

$\Downarrow$ $\lim_{x\to0}\frac{f(x)}{x}=0$

$\overset{\|}{\lim_{x\to0}\frac{x+\frac{f(x)}{x}}{x}}\Rightarrow\lim_{x\to0}\frac{f(x)}{x^2}=2$ 后续同题目①.

⑨ $\tan x=\frac{\sin x}{\cos x}=(x-\frac{x^3}{3!}+\frac{x^5}{5!}\cdots)\cdot\frac{1}{1-\frac{x^2}{2}+\frac{x^4}{4!}\cdots}=(x-\frac{x^3}{6}+\frac{x^5}{120}-\cdots)(1+\frac{x^2}{2}-\frac{x^4}{4!}+(\frac{x^2}{2}-\frac{x^4}{4!})^2+\cdots)$

$\frac{1}{1-x}=1+x+x^2+x^3+\cdots$

$=(x-\frac{x^3}{6}+\frac{x^5}{120})(1+\frac{x^2}{2}+\frac{5}{24}x^4+\cdots)$

$=x+(\frac{1}{2}-\frac{1}{6})x^3+(\frac{5}{24}-\frac{1}{12}+\frac{1}{120})x^5+o(x^5)$

$=x+\frac{1}{3}x^3+\frac{2}{15}x^5+o(x^5).$

• $\lim_{x\to0}\frac{x\tan x-\sin^2x}{x^4}=\frac{x\cdot(x+\frac{1}{3}x^3+o(x^3))-(x-\frac{x^3}{6}+o(x^3))^2}{x^4}=\lim_{x\to0}\frac{\frac{2}{3}x^4+o(x^4)}{x^4}=\frac{2}{3}$

注: 一般来说,先把能算的极限分出来: $\lim_{x\to0}\frac{x\tan x-\sin^2x}{x^4}=\lim_{x\to0}\frac{\frac{x}{\cos x}-\sin x}{x^3}\cdots\cdots$

*补充: $\sqrt{1+\sin x}$, $\sqrt{2-\cos x}$ 的Taylor公式. $(1+x)^\alpha=\binom{\alpha}{0}\cdot1+\binom{\alpha}{1}x+\binom{\alpha}{2}x^2+\cdots\cdots$

$\sqrt{1+(1-\cos x)}$ 不能 $\sqrt{2}\cdot\sqrt{1-\frac{\cos x}{2}}$ X.

• $\lim_{x\to+\infty}[(x^3-x^2+\frac{x}{2})e^{\frac{1}{x}}-\sqrt{x^6-1}]=\lim_{x\to+\infty}x^3\cdot[(1-\frac{1}{x}+\frac{1}{2x^2})e^{\frac{1}{x}}-\sqrt{1-\frac{1}{x^6}}]$. 令 $u=\frac{1}{x}$.

$=\lim_{u\to0^+}\frac{(1-u+\frac{1}{2}u^2)e^u-\sqrt{1-u^6}}{u^3}=\lim_{u\to0^+}\frac{(1-u+\frac{1}{2}u^2)(1+u+\frac{u^2}{2}+\frac{u^3}{6}+\cdots)-(1+\binom{\frac{1}{2}}{1}(-u^6)+\cdots)}{u^3}.$

⑨

$$= \lim_{u\to 0^+}\frac{\left(1+u+\frac{u^2}{2}+\frac{u^3}{6}\right)+\left(-u-u^2-\frac{u^3}{2}\right)+\left(\frac12 u^2+\frac12 u^3\right)-1+o(u^3)}{u^3} = \lim_{u\to 0^+}\frac{\frac16 u^3+o(u^3)}{u^3} = \frac16$$

$$\frac{1}{1+\xi_n^2}\left(\frac{a}{n}-\frac{a}{n+1}\right) \qquad \frac{a}{n+1}<\xi_n<\frac{a}{n}\to 0$$

⑩

(1) $$\lim_{n\to\infty} n^2\left(\arctan\frac{a}{n}-\arctan\frac{a}{n+1}\right) = \lim_{n\to\infty}\frac{1}{1+\xi_n}\lim_{n\to\infty} n^2\cdot\frac{a}{n(n+1)} = a.$$

(2) $$e = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{(n+1)!}+o\left(\frac{1}{(n+1)!}\right).$$

$$|\sin(\pi n! e)| = \left|\sin\left[\pi\left(n!+n!+\frac{n!}{2!}+\cdots+\frac{n!}{n!}\right)+\pi\left(\frac{n!}{(n+1)!}+o\left(\frac{1}{(n+1)!}\right)\right)\right]\right| = \left|\sin\left(\frac{\pi}{n+1}+o\left(\frac{1}{(n+1)!}\right)\right)\right|$$

$$\Rightarrow \lim_{n\to\infty} n|\sin(\pi n! e)| = \lim_{n\to\infty} n\sin\left[\frac{\pi}{n+1}+o\left(\frac{1}{(n+1)!}\right)\right] = \lim_{n\to\infty}\frac{n\pi}{n+1} = \pi.$$

! ⑩ (2) 的补充.

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!} + \frac{e^{\theta}}{(n+2)!}$$

$$n!e = \left(n! + n! + \frac{n!}{2!} + \cdots + \frac{n!}{n!}\right) + \frac{1}{n+1} + \frac{e^{\theta}}{(n+1)(n+2)}$$

$$n\left|\sin(\pi n! e)\right| = n\sin\left(\frac{\pi}{n+1} + \frac{e^{\theta}\pi}{(n+1)(n+2)}\right) = n\cdot\left(\frac{\pi}{n+1} + \frac{e^{\theta}\pi}{(n+1)(n+2)}\right) \to \pi$$