答案:

1.(1) 按第一行展开:

$$D_n = a\begin{vmatrix} a & b & & & \\ c & a & b & & \\ & c & a & \ddots & \\ & & \ddots & \ddots & b \\ & & & c & a \end{vmatrix} - b\begin{vmatrix} a & b & & & \\ c & a & b & & \\ & c & a & \ddots & \\ & & \ddots & \ddots & b \\ & & & c & a \end{vmatrix}$$

$$= aD_{n-1} - bcD_{n-2}$$

由特征方程: $x^2 - ax + bc = 0$ 其根为 α, β.

$$D_1 = a = \alpha + \beta \qquad D_2 = a^2 - bc = \alpha^2 + \alpha\beta + \beta^2$$

1. 若 $\alpha \neq \beta$: $D_n = A\alpha^n + B\beta^n$

求出: $$D_n = \frac{\alpha^{n-1}(D_2 - \beta D_1) - \beta^{n-1}(D_2 - \alpha D_1)}{\alpha - \beta}$$

$$= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$$

2. 若 $\alpha = \beta = \frac{a}{2}$ $D_n = (A + nB)\alpha^n$

$$= (n+1)\cdot\left(\frac{a}{2}\right)^n$$ □

(2). 按第 n 行展开:

$$D_n = 2\cos\theta\, D_{n-1} - D_{n-2}$$

特征方程: $x^2 - 2\cos\theta\, x + 1 = 0 \longrightarrow x = e^{i\theta}$ or $e^{-i\theta}$

$$\therefore D_n = Ae^{in\theta} + Be^{-in\theta} = \frac{1}{2}\left(e^{in\theta} + e^{-in\theta}\right) = \cos n\theta.$$ □

2.(1) 第$i+1$行减第i行$\cdot x_1$：

$$V_n(x_1,\cdots,x_n)=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & x_2-x_1 & \cdots & x_n-x_1 \\ \vdots & \vdots & & \vdots \\ 0 & x_2^{n-1}-x_1x_2^{n-2} & \cdots & x_n^{n-1}-x_1x_n^{n-2} \end{vmatrix}$$

$$=(x_2-x_1)\cdots(x_n-x_1)V_n(x_2,\cdots,x_n)$$

$$=\prod_{1\le i<j\le n}(x_j-x_i)$$

(2) $$\begin{vmatrix} 1 & x_1+a_{11} & x_1^2+a_{21}x_1+a_{22} & \cdots & x_1^{n-1}+a_{n-1,1}x_1^{n-2}+\cdots+a_{n-1,n-1} \\ 1 & x_2+a_{11} & x_2^2+a_{21}x_2+a_{22} & \cdots & x_2^{n-1}+a_{n-1,1}x_2^{n-2}+\cdots+a_{n-1,n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n+a_{11} & x_n^2+a_{21}x_n+a_{22} & \cdots & x_n^{n-1}+a_{n-1,1}x_n^{n-2}+\cdots+a_{n-1,n-1} \end{vmatrix}$$

$$=\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix}=\prod_{1\le i<j\le n}(x_j-x_i)$$

$\square$

3. 方法一：

$$F(t)=\sum_{(i_1,\cdots,i_n)\in S_n}\delta(i_1,\cdots,i_n)\,a_{1,i_1}(t)\,a_{2,i_2}(t)\cdots a_{n,i_n}(t)$$

$$F'(t)=\sum_{(i_1,\cdots,i_n)\in S_n}\delta(i_1,\cdots,i_n)\,a_{1,i_1}'(t)\,a_{2,i_2}(t)\cdots a_{n,i_n}(t)$$

$$+\sum_{(i_1,\cdots,i_n)\in S_n}\delta(i_1,\cdots,i_n)\,a_{1,i_1}(t)\,a_{2,i_2}'(t)\cdots a_{n,i_n}(t)$$

$$+\cdots$$

$$+\sum_{(i_1,\cdots,i_n)\in S_n}\delta(i_1,\cdots,i_n)\,a_{1,i_1}(t)\,a_{2,i_2}(t)\cdots a_{n,i_n}'(t)$$

$$=\backsim$$

方法二：直接利用多重线性映射的求导：

$$f'(x_1(t),\cdots,x_n(t))=\sum_{i=1}^{n}f(x_1(t),\cdots,x_i'(t),\cdots,x_n(t)).\qquad\square$$

4.(1) 由 Laplace 变换可得，其中第二个的系数为：

$$(-1)^{(1+2+\cdots+m)+(n+1+\cdots+n+m)}=(-1)^{mn+m(m+1)}=(-1)^{mn}.$$

(2) A 可逆：
$$\begin{vmatrix}A&B\\C&D\end{vmatrix}=\begin{vmatrix}A&B\\C-CA^{-1}A&D-CA^{-1}B\end{vmatrix}$$

$$=|A||D-CA^{-1}B|\underset{AC=CA}{=}|AD-CB|$$

(3) 假如 A 不可逆，但是 $\exists\, t\in \mathring{U}(0,\delta)$ 总有 $A+tI_n$ 可逆. why?

且：$(A+tI_n)C=C(A+tI_n)$

则：$\begin{vmatrix}A+tI_n&B\\C&D\end{vmatrix}=|(A+tI_n)D-BC|.$

∴ 令 $t\to0$ 即可得证. ← why? $\square$

5. $\lambda=0$ 时显然.

$\lambda\neq0$ 时：$\begin{vmatrix}\lambda I_m&A_{m\times n}\\B_{n\times m}&\lambda I_n\end{vmatrix}=\begin{vmatrix}\lambda I_m&A_{m\times n}\\0&\lambda I_n-BA\end{vmatrix}=\lambda^m\cdot|\lambda I_n-BA|$

$$\parallel$$

$$\begin{vmatrix}\lambda I_m-AB&0\\B_{n\times m}&\lambda I_n\end{vmatrix}=\lambda^n\cdot|\lambda I_m-AB|.$$

$\square$

6. $|A| \neq 0 \Leftrightarrow Ax=0$ 只有零解

假设存在非零解 $x_0=(b_1, b_2, \cdots, b_n)$.

不妨再设 $\max\{|b_1|, \cdots, |b_n|\} = |b_1|$

则 $|a_{11}\cdot b_1| = |a_{11}||b_1| > \sum_{i=2}^{n} |a_{1i}||b_i|$

即 $Ax_0 \neq 0$ 矛盾 □

7. 此处以 $n=3$ 为例:

$\because \deg(f_k) \leq 1$ 设 $f_k(x) = b_{k0}+b_{k1}x$

i.e. $\begin{vmatrix} b_{10}+b_{11}a_1 & b_{10}+b_{11}a_2 & b_{10}+b_{11}a_3 \\ b_{20}+b_{21}a_1 & b_{20}+b_{21}a_2 & b_{20}+b_{21}a_3 \\ b_{30}+b_{31}a_1 & b_{30}+b_{31}a_2 & b_{30}+b_{31}a_3 \end{vmatrix}$

显然思路并是不断消去 a_i 最高项，那么最后必有两列是相同的 $(\deg=0)$ □

8. $\begin{vmatrix} A & B & C & D \\ B & A & D & C \\ C & D & A & B \\ D & C & B & A \end{vmatrix} = |A+B+C+D| \begin{vmatrix} I & I & I & I \\ B & A & D & C \\ C & D & A & B \\ D & C & B & A \end{vmatrix}$

$= |A+B+C+D| \begin{vmatrix} A-B & D-B & C-B \\ D-C & A-C & B-C \\ C-D & B-D & A-D \end{vmatrix}$

$= |A+B+C+D||A+B-C-D| \begin{vmatrix} I & D-B & C-B \\ -I & A-C & B-C \\ -I & B-D & A-D \end{vmatrix}$

$= |A+B+C+D||A+B-C-D||A-B+C-D||A-B-C+D|$ □

9. 求 $|A|$ 所有代数余子式的和 $\leftrightarrow$ A^* 所有元素之和

$\leftrightarrow$ $A^* = A^{-1}$.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & \cdots & 1 & 1 & & & \\ & 1 & \cdots & 1 & & 1 & & \\ & & \ddots & \vdots & & & \ddots & \\ & & & 1 & & & & 1 \end{array}\right] \to A^{-1} = \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & \ddots & -1 \\ & & & 1 \end{bmatrix}$$

故所求值为1. □

10. 均减去第n行:

$$D_n = \begin{vmatrix} \frac{a_n - a_1}{(a_1+b_1)(a_n+b_1)} & \cdots & \frac{a_n - a_1}{(a_1+b_n)(a_n+b_n)} \\ \vdots & & \vdots \\ \frac{1}{a_n+b_1} & \cdots & \frac{1}{a_n+b_n} \end{vmatrix}$$

$$= \frac{\prod_{i=1}^{n-1}(a_n - a_i)}{\prod_{j=1}^{n}(a_n+b_j)} \begin{vmatrix} \frac{1}{a_1+b_1} & \cdots & \frac{1}{a_1+b_n} \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{vmatrix}$$

均减去第n列:

$$\downarrow$$

$$\begin{vmatrix} \frac{b_n - b_1}{(a_1+b_1)(a_1+b_n)} & \cdots & \frac{1}{a_1+b_n} \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{vmatrix}$$

$$\therefore D_n = \frac{\prod_{i=1}^{n-1}(a_n - a_i)\prod_{i=1}^{n-1}(b_n - b_i)}{\prod_{j=1}^{n}(a_n+b_j)\prod_{i=1}^{n-1}(a_i+b_n)} D_{n-1} = \frac{\prod_{i<j}(a_j - a_i)(b_j - b_i)}{\prod_{i,j}(a_i + b_j)}$$

□

11. $\cos\theta = \cos\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ $\longrightarrow$ $\cos n\theta = f(\cos\theta)$

$\vdots$ $= \underline{2^n \cos^n\theta} + \square \cos^{n-1}\theta$

$+ \cdots + \square\cos\theta + \square$

$\cos n\theta = 2\cos\theta \cdot \cos(n-1)\theta - \cos(n-2)\theta$

故消去低次项后，又变为 Vandermonde 行列式：

$$D_n = \begin{vmatrix} 1 & \cos\theta_1 & 2\cos^2\theta_1 & \cdots & 2^{n-1}\cos^{n-1}\theta_1 \\ 1 & \cos\theta_2 & 2\cos^2\theta_2 & \cdots & 2^{n-1}\cos^{n-1}\theta_2 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \cos\theta_n & 2\cos^2\theta_n & \cdots & 2^{n-1}\cos^{n-1}\theta_n \end{vmatrix}$$

$$= 2^{\frac{n(n-1)}{2}} \prod_{1\le i<j\le n} (\cos\theta_j - \cos\theta_i).$$

$\square$

12. (1) 方法一：

可以硬算，得：$\det(A) = (af - be + cd)^2$

方法二：

Pfaffian 数：

$$pf(A)\, e_1\wedge e_2\wedge e_3\wedge e_4 = \frac{1}{2!}\Lambda^2(a e_1\wedge e_2 + b e_1\wedge e_3 + c e_1\wedge e_4$$
$$+ d e_2\wedge e_3 + e\, e_2\wedge e_4 + f e_3\wedge e_4)$$
$$= (af - be + cd)\, e_1\wedge e_2\wedge e_3\wedge e_4$$

$\therefore\ pf(A) = af - be + cd$

故 $\det(A) = (af - bc + cd)^2$

$\square$

(2) 证明偶数(2n)阶反对称阵的行列式为完全平方:

用归纳法.

$2n=2$ 时: $\begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix} = a^2$ 成立.

假设对 $2(n-1)$ 成立, 证明 $2n$ 时的情况:

不妨假设 $a_{12}\neq 0$, 则使用初等行列变换使得:

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ -a_{12} & 0 & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ -a_{1n} & -a_{2n} & -a_{3n} & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & b_2 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & b_n & 0 & \cdots & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & a_{12} & 0 & \cdots & 0 \\ -a_{12} & * & * & \cdots & * \\ 0 & * & & & \\ \vdots & \vdots & & B & \\ 0 & * & & & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ & 1 & b_2 & \cdots & b_n \\ & & 1 & \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & 1 \end{pmatrix}^{-1}$$

必然互为转置, why?

$\therefore \det(A_{2n\times 2n}) = a_{12}^2 \cdot \det(B_{2n-2\times 2n-2})$ 是完全平方.

↓归纳假设

完全平方

从而由数学归纳法可证. □

例:

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & -2 & 1 & \\ & -3 & & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 4 & 5 \\ -2 & -4 & 0 & 6 \\ -3 & -5 & -6 & 0 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & -2 & -3 \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 4 & 5 \\ 0 & -4 & -8 & -4 \\ 0 & -5 & -18 & -15 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & -2 & -3 \\ & & 1 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 4 & 5 \\ 0 & -4 & \boxed{\begin{matrix} 0 & 8 \\ 8 & 0 \end{matrix}} \\ 0 & -5 & & \end{bmatrix}$$

13. 方法一：Pfaffian 数：

两边取 pf 即有：

$$\det(A)\cdot pf\begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix} = \underbrace{pf\begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}}_{\neq 0.}$$

$\therefore \det(A)=1.$

在 $\mathbb{C}_{2n\times 2n}$ 中也一样.

方法二：扰动法：

将 A 分块为 $\begin{pmatrix} M & N \\ P & Q \end{pmatrix}$，故条件即为：

$$\begin{pmatrix} MN^T-NM^T & MQ^T-NP^T \\ PN^T-QM^T & PQ^T-QP^T \end{pmatrix} = \begin{pmatrix} & E_n \\ -E_n & \end{pmatrix}$$

i.e. $QM^T-PN^T=I_n$ & $MN^T=NM^T$ & $PQ^T=QP^T$

若 M 可逆，则：$|A| = \begin{vmatrix} M & N \\ O & Q-PM^{-1}N \end{vmatrix}$

$$= |Q-PM^{-1}N||M^T|$$

$$= |QM^T-PN^T| = |I_n| = 1.$$

若 M 不可逆，需保证扰动后仍有 $M(t)N^T=NM(t)^T$ 才行.

故不可令：$M(t)=M+tI_n$

但是有结论：

利用 Jordan 分解

不妨 $N=P\begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}P^{-1}$

$= \underbrace{P\begin{bmatrix} & & 1 & \lambda \\ & \iddots & \iddots & \\ 1 & \lambda & & \end{bmatrix}P^T}_{S_1} \cdot \underbrace{(P^{-1})^T\begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}P^{-1}}_{S_2}$

N 必然可写成两个对称阵的乘积

i.e. $N=S_1S_2$ 其中 $S_1^T=S_1$，$S_2^T=S_2$ 且 S_2 可逆.

从而扰动为：$M(t)=M+tS_2^{-1}$

$$|A(t)| = \begin{vmatrix} M(t) & N \\ P & Q \end{vmatrix} = |I_n+tQS_2^{-1}| \to 1 \quad (t\to 0)$$

□